

Cylindric versions of specialised Macdonald functions and a deformed Verlinde algebra

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Abstract

We define cylindric generalisations of skew Macdonald functions $P_{\lambda/\mu}(q, t)$ when either $q = 0$ or $t = 0$. Fixing two integers $n > 2$ and $k > 0$ we shift the skew diagram λ/μ , viewed as a subset of the two-dimensional integer lattice, by the period vector $(n, -k)$. Imposing a periodicity condition one defines cylindric skew tableaux and associated weight functions. The resulting weighted sums over these cylindric tableaux are symmetric functions. They appear in the coproduct of a commutative Frobenius algebra which is a particular quotient of the spherical Hall algebra. We realise this Frobenius algebra as a commutative subalgebra in the endomorphisms over a $U_q \widehat{\mathfrak{sl}}(n)$ Kirillov-Reshetikhin module. Acting with special elements of this subalgebra, which are noncommutative analogues of Macdonald polynomials, on a highest weight vector, one obtains Lusztig's canonical basis. In the limit $q = t = 0$, this Frobenius algebra is isomorphic to the $\widehat{\mathfrak{sl}}(n)$ Verlinde algebra at level k , i.e. the structure constants become the $\widehat{\mathfrak{sl}}(n)_k$ Wess-Zumino-Novikov-Witten fusion coefficients. Further motivation comes from exactly solvable lattice models in statistical mechanics: the cylindric Macdonald functions discussed here arise as partition functions of so-called vertex models obtained from solutions to the Yang-Baxter equation. We show this by stating explicit bijections between cylindric tableaux and lattice configurations of non-intersecting paths. Using the algebraic Bethe ansatz the idempotents of the Frobenius algebra are computed.

1 Introduction

The fusion or Verlinde ring \mathcal{V}_k , $k \in \mathbb{Z}_{\geq 0}$ of a Kac Moody algebra $\hat{\mathfrak{g}}$ is a particular finite-dimensional quotient of the Grothendieck ring $\text{Rep } \mathfrak{g} = \bigoplus_{\lambda \in \mathcal{P}^+} \mathbb{Z}[\pi_\lambda]$ (with product \otimes), where \mathfrak{g} is the corresponding (non-affine) semi-simple Lie algebra, \mathcal{P}^+ is the set of dominant integral weights and $[\pi_\lambda]$ stands for the isomorphism class of the irreducible representation π_λ with highest weight λ . Given a non-negative integer k define \mathcal{I}_k to be the ideal generated by elements of the form $[\pi_\lambda] - (-1)^{\ell(w)}[\pi_{w \circ \lambda}]$ where w is an element in the affine Weyl group \tilde{W} and $w \circ \lambda = w(\lambda + \rho) - \rho$ denotes the non-affine part of the weight obtained under the shifted level- k action of \tilde{W} with ρ being the Weyl vector. For instance, in the case of $\mathfrak{g} = \mathfrak{sl}(n)$ the action of the simple Weyl reflections is detailed in equation (2.5) in the text. The Verlinde algebra is then defined as $\mathcal{V}_k := \text{Rep } \mathfrak{g} / \mathcal{I}_k$; this is in essence the celebrated Kac-Walton formula [33] [75]. The structure constants of \mathcal{V}_k are known to coincide with the fusion coefficients in Wess-Zumino-Novikov-Witten (WZNW) conformal field theory, dimension of moduli spaces of generalised θ -functions and multiplicities of tilting modules of quantum groups at roots of unity.

1.1 Review of previous results

In this article we will only consider the simplest case when $\mathfrak{g} = \mathfrak{sl}(n)$ or $\mathfrak{gl}(n)$ with Weyl group $W = \mathfrak{S}_n$, the symmetric group. There is a ring isomorphism $\chi : \text{Rep } \mathfrak{gl}(n) \rightarrow \mathbb{Z}[x_1, \dots, x_n]^{\mathfrak{S}_n}$ which maps each isomorphism class $[\pi_\lambda]$ onto its Weyl character which can be identified with the Schur function s_λ and we have $s_\mu s_\nu = \sum_{\lambda \in \mathcal{P}_n^+} c_{\mu\nu}^\lambda s_\lambda$ with $c_{\mu\nu}^\lambda$ being the famous Littlewood-Richardson coefficients. In what follows it will be important to note that the ring of symmetric functions $\mathbb{Z}[x_1, \dots, x_n]^{\mathfrak{S}_n}$ can be turned into an infinite-dimensional bialgebra [23] [76] (which for convenience we define over \mathbb{C}) with coproduct $\Delta s_\lambda = \sum_{\mu \in \mathcal{P}^+} s_{\lambda/\mu} \otimes s_\mu$ where $s_{\lambda/\mu} = \sum_{\nu \in \mathcal{P}_n^+} c_{\mu\nu}^\lambda s_\nu$ is the skew Schur function.

Following [25], [28] define $\mathcal{I}_k = \langle s_{(1^n)} - 1, s_{(k+1)}, s_{(k+2)}, \dots, s_{(k+n-1)} \rangle$.

Theorem 1.1 (Gepner, Goodman-Wenzl) *The map $[\pi_\lambda] \mapsto [s_\lambda] := s_\lambda + \mathcal{I}_k$ defines a ring isomorphism $\chi_k : \mathcal{V}_k \rightarrow \mathbb{Z}[x_1, \dots, x_n]^{\mathfrak{S}_n} / \mathcal{I}_k$.*

Define a non-degenerate bilinear form $\eta([\pi_\lambda], [\pi_\mu]) = \delta_{\lambda\mu^*}$ on the Verlinde algebra $\mathcal{V}_k \otimes_{\mathbb{Z}} \mathbb{C}$, where $\lambda, \mu \in \mathcal{P}_{n,k}^+$ are dominant integral weights at level k and μ^* denotes the contragredient weight of μ . Then $(\mathcal{V}_k \otimes_{\mathbb{Z}} \mathbb{C}, \eta)$ is a (finite-dimensional) commutative Frobenius algebra. This fact is not often mentioned in the literature, but it will motivate our definition of a deformed Verlinde algebra below.

In [40] it was shown that there exists an alternative, combinatorial description of the Verlinde algebra which employs a local affine version of the plactic algebra [49] in the Robinson-Schensted-Knuth correspondence. Each dominant integral weight $\lambda \in \mathcal{P}_{n,k}^+$ at level k corresponds to a unique composition $m(\lambda) = (m_1, \dots, m_n) \in \mathbb{Z}_{\geq 0}^n$ where $m_i = m_i(\lambda)$ is the multiplicity of the part i in the conjugate partition λ' for $i = 1, \dots, n-1$ and $k = \sum_{i=1}^n m_i$. Interpret each $m(\lambda)$ as a particle configuration on the $\widehat{\mathfrak{sl}}(n)$ Dynkin diagram where $m_i(\lambda)$ particles are sitting at the i th node; see Figure 1.1 for a simple example. Define maps $\beta_i^*, \beta_i : \mathcal{P}_{n,k}^+ \rightarrow \mathcal{P}_{n,k \pm 1}^+$ which increase and decrease the number of particles at node i by one, respectively.

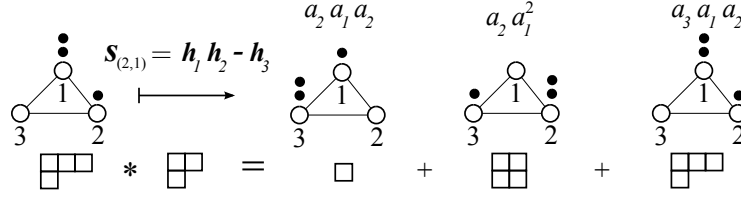


Figure 1.1: Graphical depiction of the combinatorial fusion product for $n = k = 3$. The start configuration corresponds to the weight $\lambda = 2\omega_1 + \omega_2$. Acting with the affine plactic Schur polynomial $s_{(2,1)}$ yields three nonzero terms; each is obtained as a sequence of 3 hopping moves written as monomials in the a_i 's (top). The bottom line shows the fusion product where columns of height 3 have been dropped from the Young diagrams which represent the weights.

The *affine plactic algebra* is then generated by the maps $a_i = \beta_{i+1}^* \beta_i$ which move one particle from node i to node $i + 1$ with $i \in \mathbb{Z}_n$. The directed coloured graph obtained from setting $\lambda \xrightarrow{i} \mu$ if $a_i m(\lambda) = m(\mu)$ matches the Kirillov-Reshetikhin crystal graph $B^{1,k}$ of the quantum enveloping algebra $U_q \widehat{\mathfrak{sl}}(n)$. Define the *affine plactic Schur polynomial* [40,41] as $s_\lambda := \det(h_{\lambda_i - i + j})_{1 \leq i, j \leq n}$ with $h_r = \sum_{\mu \vdash r} (\beta_1^*)^{\mu_n} a_1^{\mu_1} \cdots a_{n-1}^{\mu_{n-1}} \beta_n^{\mu_n}$. The polynomial s_λ is well-defined, since one can show that $h_r h_{r'} = h_{r'} h_r$ for all $r, r' \in \mathbb{Z}_{\geq 0}$.

Theorem 1.2 (Korff-Stroppel) *Consider the free abelian group $\mathbb{Z}\mathcal{P}_{n,k}^+$ with respect to addition. Introduce the product $\lambda \circledast \mu := s_\lambda \mu$, then $(\mathbb{Z}\mathcal{P}_{n,k}^+, \circledast)$ is canonically isomorphic to the Verlinde ring \mathcal{V}_k .*

Note that s_λ specialises to the finite, non-affine plactic Schur polynomial of Fomin and Greene [19] when setting formally $a_n = 0$; see also the construction of noncommutative Schur's P, Q -functions using a shifted plactic monoid in [69]. The combinatorics of these constructions is less involved than the one of the affine polynomials. Other approaches to noncommutative symmetric functions can be found in [24] and, in particular, noncommutative Hall-Littlewood functions have been discussed in [31], [59].

Define the two-sided ideal $\mathcal{I}'_k = \langle s_{(n)} - 1, s_{(n+1)}, s_{(n+2)}, \dots, s_{(n+k-1)} \rangle$.

Theorem 1.3 (Korff-Stroppel) *The map $[\pi_\lambda] \mapsto [s_{\lambda'}] := s_{\lambda'} + \mathcal{I}'_k$, where λ' is the conjugate partition, defines a ring isomorphism $\chi'_k : \mathcal{V}_k \rightarrow \mathbb{Z}[x_1, \dots, x_k]^{\mathfrak{S}_k} / \mathcal{I}'_k$.*

This result [40, Theorem 1.3] is intimately linked to a quantum integrable system, the so-called phase model, which has been considered in [7]. Interpret the complex linear span $\mathbb{C}\mathcal{P}_{n,k}^+$ as the state space of a discrete quantum mechanical system and $H_r^\pm = s_{(r)} \pm s_{(n-r)}$ as its set of commuting quantum Hamiltonians. Computing the common eigenbasis of the latter via the so-called *Bethe ansatz* or the *quantum inverse scattering method* leads to the coordinate ring of a finite 0-dimensional affine variety, the solutions to the so-called Bethe ansatz equations. This coordinate ring is the quotient ring $\mathbb{Z}[x_1, \dots, x_k]^{\mathfrak{S}_k} / \mathcal{I}'_k$ and the so-called Bethe states, the eigenstates of the quantum Hamiltonians, are the idempotents of the Verlinde algebra [40] [42].

1.2 Deformed fusion matrices and canonical bases

The main result of this article is that the combinatorial description of the Verlinde algebra in terms of affine plactic Schur polynomials and Kirillov-Reshetikhin crystal graphs can be lifted to the quantum affine algebra $U'_q \widehat{\mathfrak{gl}}(n)$ using Lusztig's canonical basis [51] [52], which is the same as Kashiwara's lower crystal basis [35]. This induces in a natural way a product on the corresponding Kirillov-Reshetikhin module which then becomes a commutative Frobenius algebra.

The central algebraic object which we are going to employ is the n -fold tensor product $\mathcal{H}_q^{\otimes n}$ of the q -oscillator or Heisenberg algebra \mathcal{H}_q whose generators will be realised as maps $\beta_i^*, \beta_i : \mathbb{C}(q)\mathcal{P}_{n,k}^+ \rightarrow \mathbb{C}(q)\mathcal{P}_{n,k\pm 1}^+$ which generalise the maps mentioned previously in the context of the combinatorial Verlinde algebra. We will show that there exists an algebra homomorphism $U'_q \widehat{\mathfrak{gl}}(n) \rightarrow \mathcal{H}_q^{\otimes n}$ which allows one to pull the n -fold tensor product of any \mathcal{H}_q -module back to the quantum affine algebra. The linear span of the particle configurations on the $\widehat{\mathfrak{sl}}(n)$ Dynkin diagram discussed earlier corresponds to the infinite-dimensional highest weight module known as Fock space $\mathcal{F}^{\otimes n} \cong \bigoplus_{k \geq 0} \mathbb{C}(q)\mathcal{P}_{n,k}^+$. Denote by $S^k(V)$ the k^{th} divided power in the quantum symmetric tensor algebra of the vector representation V of $U_q \mathfrak{gl}(n)$ and let ω_1 be the first fundamental affine weight of $\widehat{\mathfrak{sl}}(n)$.

Proposition 1.4 *There exists a $U'_q \widehat{\mathfrak{sl}}(n)$ -module isomorphism $\mathcal{F}^{\otimes n} \cong \bigoplus_{k \geq 0} W^{1,k}$, where $W^{1,k}$ is the Kirillov-Reshetikhin module $W(k\omega_1)$. When restricting to the finite algebra $U_q \mathfrak{gl}(n)$ one obtains the module isomorphism $\mathcal{F}^{\otimes n} \cong S(V) := \bigoplus_{k \geq 0} S^k(V)$.*

Similar to the case of the Verlinde algebra, we consider the (noncommutative) subalgebra $\subset \mathcal{H}_q^{\otimes n}$ which is generated by the alphabet $\{a_i = \beta_{i+1}^* \beta_i : i \in \mathbb{Z}_n\}$. The latter corresponds to the images of the products $\{K_1 F_1, \dots, K_n F_n\}$ of quantum group Chevalley generators under the above homomorphism $U_q \widehat{\mathfrak{gl}}(n) \rightarrow \mathcal{H}_q^{\otimes n}$. In particular they obey

$$\begin{aligned} a_i a_j &= a_j a_i, & |i - j| > 1, \\ a_{i+1} a_i^2 + q^2 a_i^2 a_{i+1} &= (1 + q^2) a_i a_{i+1} a_i, \\ a_{i+1}^2 a_i + q^2 a_i a_{i+1}^2 &= (1 + q^2) a_{i+1} a_i a_{i+1}, & i, j \in \mathbb{Z}_n. \end{aligned} \quad (1.1)$$

These identities are simply the quantum Serre relations of $U_q \widehat{\mathfrak{gl}}(n)$ rewritten in the generators $K_i F_i$. In the crystal limit $q = 0$ (1.1) are the Knuth relations of the (local) affine plactic algebra considered in [40] and we have now the following generalisation of the affine plactic Schur polynomials.

Let $|k^n\rangle$ denote the $U_q \widehat{\mathfrak{sl}}(n)$ highest weight vector in $S^k(V)$ and denote by $B_{n,k} = \{|\lambda\rangle : \lambda \in \mathcal{P}_{n,k}^+\} \subset S^k(V)$ the canonical basis. The precise definition of the basis vectors $|\lambda\rangle$ will be given in the text.

Theorem 1.5 (deformed fusion matrices) *There exists a set of commuting elements $\mathbf{B}_n := \{Q'_\lambda : \lambda_1 \geq \dots \geq \lambda_n, \lambda_i \in \mathbb{Z}_{\geq 0}\} \subset \mathcal{H}_q^{\otimes n}$, polynomial in the a_i 's, such that*

$$|\mu\rangle \otimes |\nu\rangle := Q'_\mu |\nu\rangle, \quad \mu, \nu \in \mathcal{P}_{n,k}^+ \quad (1.2)$$

defines a commutative Frobenius algebra $\mathfrak{F}_{n,k} = (\mathbb{C}(q)\mathcal{P}_{n,k}^+, \otimes)$. The unit is given by the highest weight vector $|k^n\rangle$, that is $Q'_\lambda |k^n\rangle = |\lambda\rangle$.

Thus, the set $B_n \subset \mathcal{H}_q^{\otimes n}$ generates the canonical basis in each $S^k(V)$ when acting on the respective highest weight vector. The polynomials Q'_λ in the generators a_i exhibit a particularly nice structure which allows one to identify them as noncommutative analogues of Macdonald functions, where one parameter is set to zero. We refer to $\{Q'_\lambda\}$ as deformed fusion matrices since setting formally $q = 0$ in (1.1) one recovers the combinatorial ring in Theorem 1.2.

To put these findings further into perspective we recall that Lusztig's geometric construction of the canonical basis $\mathfrak{B} \subset U_q \mathfrak{n}^-$ focusses on polynomials in the F_i 's, that is, one considers $U_q \mathfrak{n}^-$ instead of $U_q \mathfrak{b}^-$. Certain special elements X, Y in the *dual* canonical basis \mathfrak{B}^* are known to quasi-commute, $XY = qYX$; this was conjectured in [5] and proven for semi-simple quantum algebras in [60] using Ringel's Hall algebra approach. For $U_q \widehat{\mathfrak{gl}}(n)$ with q a root of unity the canonical basis is known to be linked to the Ringel-Hall algebra of the cyclic quiver [64].

1.3 The deformed Verlinde algebra: Demazure characters

Denote by $Q_\lambda(q, t) = b_\lambda(q, t)P_\lambda(q, t)$ the celebrated Macdonald functions, where $b_\lambda(q, t)$ is some normalisation factor; details will be provided in the text. Consider the limit $P_\lambda := P_\lambda(0, t)$, $Q_\lambda := Q_\lambda(0, t) \in \mathbb{C}(t)[x_1, \dots, x_k]^{\mathfrak{S}_k}$ which are the celebrated Hall-Littlewood functions. Then one has the product expansion

$$P_\mu P_\nu = \sum_{\lambda \in \mathcal{P}_k^+} f_{\mu\nu}^\lambda(t) P_\lambda, \quad (1.3)$$

where the $f_{\mu\nu}^\lambda(t)$ are the structure constants of Hall's algebra or the spherical Hecke algebra; see [53] and references therein.

Define the two-sided ideal

$$\mathcal{I}'_k = \langle Q_{(n)} + t^k - 1, Q_{(n+1)} + t^k \bar{Q}_1, \dots, Q_{(n+k-1)} + t^k \bar{Q}_{(k-1)} \rangle,$$

where $\bar{Q}_\lambda = Q_\lambda(0, t^{-1})$ and let $\mathbb{k} = \mathbb{C}\{\{t\}\}$ be the field of formal Puiseux series. The extension of the base field to \mathbb{k} is required to construct the idempotents of $\mathfrak{F}_{n,k}$. The following statement is the analogue of Theorem 1.3.

Theorem 1.6 (deformed Verlinde algebra) *For $t = q^2$ the map $|\lambda\rangle \mapsto [P_{\lambda'}]$ is an algebra isomorphism $\mathfrak{F}_{n,k} \otimes \mathbb{k} \cong \mathbb{k}[x_1, \dots, x_k]^{\mathfrak{S}_k} / \mathcal{I}'_k$.*

The analogue of Theorem 1.1 for $\mathfrak{F}_{n,k}$ is currently missing. However, it is more natural to consider the deformed fusion product (1.2) as a modification of the product

$$Q'_\mu Q'_\nu = \sum_{\lambda \in \mathcal{P}_n^+} f_{\mu'\nu'}^{\lambda'}(t) Q'_{\lambda'}, \quad (1.4)$$

where $Q'_\lambda := Q_\lambda(t, 0) \in \mathbb{C}(t)[x_1, \dots, x_n]^{\mathfrak{S}_n}$ is now the complementary limit of Macdonald functions and λ' denotes the conjugate partition. In the projective limit of ∞ -many variables there exists a bialgebra automorphism $\omega_t : P_\lambda(0, t) \mapsto Q_{\lambda'}(t, 0)$ which is simply the known duality relation of Macdonald functions

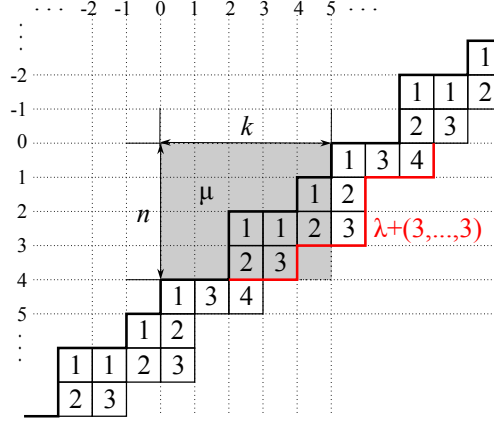


Figure 1.2: Example of a cylindric tableau for $n = 4$ and $k = 5$. Set $\lambda = (5, 3, 3, 1)$, $\mu = (5, 4, 2, 2)$ and $d = 3$.

[53, VI.5, Equation (5.1), p327] when one of the parameters is set to zero. The dual Macdonald function P'_λ can be identified with characters of so-called Demazure modules related to $\widehat{\mathfrak{sl}}(n)$ [63] (see also [32] for other Kac-Moody algebras) and [47]. Define a modified product

$$Q'_\mu * Q'_\nu := \sum_{\lambda \in \mathcal{P}_{n,k}^+} N_{\mu'\nu'}^{\lambda'}(t) Q'_\nu, \quad (1.5)$$

where $N_{\mu'\nu'}^{\lambda'}(t)$ is defined through $[P_{\mu'} P_{\nu'}] = \sum_{\lambda \in \mathcal{P}_{n,k}^+} N_{\mu'\nu'}^{\lambda'}(t) [P_{\lambda'}]$ in the quotient $\mathbb{k}[x_1, \dots, x_k]^{\mathfrak{S}_k} / \mathcal{I}'_k$. The coproduct of the resulting Frobenius algebra leads to a cylindric generalisation $P'_{\lambda/d/\mu} = \sum_{\lambda \in \mathcal{P}_{n,k}^+} N_{\mu'\nu'}^{\lambda'}(t) P'_\nu$ of the skew Macdonald function $P'_{\lambda/\mu} := P_{\lambda/\mu}(t, 0) = \sum_\nu f_{\mu'\nu'}^{\lambda'}(t) P'_\nu$. We will define $P'_{\lambda/d/\mu}$ explicitly as weighted sum over cylindric Young tableaux (also called reversed cylindric plane partitions). The latter were first considered in [27] and are maps $T : \lambda/d/\mu \rightarrow \mathbb{N}$, where $\lambda/d/\mu$ denotes a cylindric skew diagram which can be seen as set of points in \mathbb{Z}^2 obtained by a periodic continuation of the ordinary skew diagram $(\lambda_1 + d, \dots, \lambda_n + d)/\mu$ with respect to the period vector $\Omega = (n, -k)$; see Figure 1.2 for an example. We believe the cylindric skew Macdonald functions $P'_{\lambda/d/\mu}$ to be of interest because of the following observations.

Firstly, at $t = 0$ they yield cylindric Schur functions $s_{\lambda/d/\mu} = \sum_{\lambda \in \mathcal{P}_{n,k}^+} N_{\mu\nu}^\lambda(0) s_\nu$ where $N_{\mu\nu}^\lambda(0)$ are the WZNW fusion coefficients. These cylindric Schur functions arise from the coproduct of the Verlinde algebra seen as Frobenius algebra. Similar cylindric Schur functions appear in the context of the small quantum cohomology ring of the Grassmannian [56] (see also [54]). Furthermore, cylindric Young diagrams and tableaux occur in the representation theory of double affine Hecke algebras [70].

Secondly, we have the following observation:

Conjecture 1.7 *The polynomials $K_{\nu', \lambda'/d/\mu'}(t)$ defined through the expansion*

$$P'_{\lambda/d/\mu}(x_1, \dots, x_{n-1}; t) = \sum_{\nu \in \mathcal{P}_{n,k}^+} K_{\nu', \lambda'/d/\mu'}(t) s_\nu(x_1, \dots, x_{n-1}), \quad (1.6)$$

where s_ν is the Schur function, have always non-negative coefficients.

This conjecture is currently based on numerical computations, but as we will explain in the text, for an appropriate choice of μ these polynomials specialise to the celebrated Kostka-Foulkes polynomials $K_{\nu',\lambda'}(t)$ and the known expansion $P'_\lambda(t) = \sum_\mu K_{\mu',\lambda'}(t)s_\mu$ has representation theoretic interpretations: setting $t = 1$ the Kostka numbers $K_{\nu',\lambda'}(1)$ are multiplicities of finite-dimensional $\mathfrak{sl}(n)$ -modules in the Demazure module corresponding to P'_λ [63, Remark after Theorem 8]. In certain cases this result can be generalised to arbitrary t : the function $P'_\lambda(t)$ can be identified as the graded character of Demazure modules of the current algebra $\mathfrak{sl}(n) \otimes \mathbb{C}[t]$ and the coefficient of t^r in the Kostka polynomial $K_{\lambda',\mu'}(t)$ provides the dimension of subspaces of degree r in Feigin-Loktev fusion products. This has been conjectured in [17] and proved in [11, Corollary 1.5.2]; see also [47, Theorem 5.2], [29] and [67] for a computation of the graded characters using crystal bases. Results on Demazure modules and fusion products related to Lie algebras other than type A can be found in [20]. It would be desirable to find similar representation theoretic interpretations of (1.6) proving the above conjecture.

In this context we also mention that it has been shown in [2] that graded multiplicities in Feigin-Loktev fusion products of Kirillov-Reshetikhin modules are related to the generalized Kostka-Foulkes polynomials introduced in [68] and [37]. Related deformations of fusion coefficients can be found in [18] and [65, 66]. However, the deformed fusion coefficients in *loc. cit.* specialise to the known fusion coefficients at $q = 1$ instead of $q = 0$ and appear to be different from the structure constants of the Frobenius algebra discussed in this article. Also, there have been q -deformed versions of the Virasoro algebra suggested [3]; at the moment the connection between these constructions and the one in this article is unclear.

1.4 Yang-Baxter algebras and quantum integrable systems

One of the novel aspects of the combinatorial description of the Verlinde algebra in [40–42] is its identification with the commutative algebra generated by the Hamiltonians or integrals of motion of a quantum integrable system. Here we show that these findings generalize from the simple combinatorial phase model considered in the case of the Verlinde algebra to a genuinely strongly-correlated quantum many body system, the so-called q -boson model [7]. The quantum Hamiltonians generate a commutative Frobenius algebra and the latter are in one-to-one correspondence with two-dimensional topological field theories; we will address this aspect in the conclusions and connect it with recent developments in four-dimensional supersymmetric $N = 2$ gauge theories [57].

Central starting point for the algebraic formulation of a quantum integrable model are solutions to the quantum Yang-Baxter equation,

$$R_{12}(x, y)L_{13}(x)L_{23}(y) = L_{23}(y)L_{13}(x)R_{12}(x, y) . \quad (1.7)$$

The latter is an identity in $\text{End}(V(x) \otimes V(y) \otimes \mathcal{H}_q)$ where V is some complex vector space, $V(x) := \mathbb{C}[[x]] \otimes V$ and it is understood that $R(x, y) : V(x) \otimes V(y) \rightarrow V(x) \otimes V(y)$ and $L(x) : V(x) \otimes \mathcal{H}_q \rightarrow V(x) \otimes \mathcal{H}_q$ only act nontrivially in those factors of $V(x) \otimes V(y) \otimes \mathcal{H}_q$ indicated by the lower indices when numbering the spaces from left to right with 1, 2, 3. Here x, y are some formal variables which - when evaluated in the complex numbers - are called *spectral parameters*.

One can interpret the relation (1.7) as the definition of a subalgebra in \mathcal{H}_q which is called the *Yang-Baxter algebra*. The latter comes naturally equipped with a coproduct $\Delta L(x) = L_{02}(x)L_{01}(x)$ and one is led to consider the monodromy matrix $T(x) = L_{0,n}(x)L_{0,n-1}(x) \cdots L_{0,1}(x) \in \text{End}(V(x) \otimes \mathcal{H}_q^{\otimes n})$, where the first index 0 now refers to the factor $V(x)$ and the second numbers the copies of \mathcal{H}_q in $\mathcal{H}_q^{\otimes n}$. Taking the formal partial trace over V one obtains the current operator $\mathcal{O}(x) = \sum_{r \geq 0} x^r \mathcal{O}_r = \text{Tr}_V T(x)$ with $\mathcal{O}_r \in \mathcal{H}_q^{\otimes n}$ and it follows from the Yang-Baxter equation that $\mathcal{O}(x)\mathcal{O}(y) = \mathcal{O}(y)\mathcal{O}(x)$, $\forall x, y$. These are Baxter's ‘commuting transfer matrices’ and their matrix elements $\langle \lambda | \mathcal{O}(x_1) \cdots \mathcal{O}(x_\ell) | \mu \rangle$ in the Fock space $\mathcal{F}^{\otimes n}$ can be interpreted as partition functions of an exactly-solvable lattice model in statistical mechanics with periodic boundary conditions. Alternatively, one can interpret the coefficients $\{\mathcal{O}_r\}$ as the commuting Hamiltonians of a quantum integrable model. Both points of view are important.

Statistical mechanics. We will discuss two solutions (R, L) and (R', L') to the Yang-Baxter equation (1.7) setting $V = \mathbb{C}^2$ and $V = \mathcal{F}$, the infinite-dimensional Fock space of the q -Heisenberg algebra \mathcal{H}_q . The solution for $V = \mathbb{C}^2$ has been obtained previously [7], the other one is new. We will show that the resulting current operators $\mathcal{O} = \mathbf{E}$ and $\mathcal{O}' = \mathbf{G}'$ can be interpreted as noncommutative analogues of generating functions for elementary symmetric, $\mathcal{O}_r = \mathbf{e}_r$, and multivariate Rogers-Szegő polynomials, $\mathcal{O}'_r = \mathbf{Q}'_{(r)}$, in the alphabet $\{a_1, \dots, a_n\}$, respectively. They satisfy an analogue of Baxter's famous TQ -relation for the six and eight-vertex model [4]. The corresponding partition functions $\langle \lambda | \mathcal{O}(x_1) \cdots \mathcal{O}(x_\ell) | \mu \rangle$ are symmetric functions, since the current operators or transfer matrices commute. Stating explicit bijections between lattice configurations of the associated statistical mechanics models and cylindric Young tableaux we show that they yield cylindric skew Macdonald functions $Q_{\lambda'/d/\mu'}$ and $P'_{\lambda'/d/\mu} = \sum_{\lambda \in \mathcal{P}_{n,k}^+} N_{\mu'\nu'}^{\lambda'}(t) P'_\nu$ mentioned earlier. The ordinary skew Macdonald functions $Q_{\lambda'/\mu'} = Q_{\lambda'/\mu'}(0, t)$ and $P'_{\lambda'/\mu} = P_{\lambda/\mu}(t, 0)$ are obtained as special cases for $d = 0$ which corresponds to open boundary conditions on the lattice. Another instance where $P_\lambda(t, 0)$ has occurred in the context of integrable systems is in the case of the q -deformed Toda chain where they appear as eigenfunctions of the quantum Hamiltonians [26]. Here the eigenfunctions of the Hamiltonians are instead Hall-Littlewood functions; compare with [72].

Quantum integrable models. The other perspective motivated by physics is to interpret the coefficients $\{\mathcal{O}_r\}$ as quantum Hamiltonians and the canonical basis vectors in $S^k(V)$ as quantum particle configurations on a ring-shaped lattice, analogous to the discussion of the Verlinde algebra above. The two solutions (R, L) and (R', L') will yield the Hamiltonians of the so-called q -boson model and the discrete Laplacians introduced in [73] for a discrete version of the quantum nonlinear Schrödinger model. Mathematically, the $\{\mathcal{O}_r\}$ will generate the Frobenius algebra $\mathfrak{F}_{n,k}$; they can be thought of as the basic building blocks of the deformed fusion matrices in (1.2). Computing the eigenbasis $\{\mathbf{e}_\lambda\}$ of the quantum Hamiltonians $\{\mathcal{O}_r\}$ in the k -particle space, via the so-called *algebraic Bethe ansatz*, we will obtain the idempotents of the Frobenius algebra, $\mathbf{e}_\lambda * \mathbf{e}_\mu = \delta_{\lambda\mu} \mathbf{e}_\lambda$ with $1 = \sum_\lambda \mathbf{e}_\lambda$. This is known as *Peirce decomposition* in the literature [55] and requires a novel algebro-geometric proof of completeness of the Bethe ansatz for this model, which we state in Section 7.

1.5 Outline of the article

The following list summarises the discussion and results contained in each section of this article.

Section 2. We introduce the necessary conventions and algebraic notions (the extended affine symmetric group, the weight lattice, the affine Hecke algebra, the quantum affine algebra $U'_q\widehat{\mathfrak{gl}}(n)$, Macdonald functions) which we need to keep this article self-contained.

Section 3. We discuss the central algebraic object, the q -boson algebra and its Fock space representation. We state the algebra homomorphism with the quantum affine algebra $U'_q\widehat{\mathfrak{gl}}(n)$ and prove the module isomorphism with the Kirillov-Reshetikhin module. In particular, we identify the basis in the Fock space with Lusztig's canonical basis in the quantum algebra module. Then we introduce several solutions to the Yang-Baxter equation in terms of the q -boson algebra and use them to introduce analogues of Macdonald polynomials in a noncommutative alphabet.

Section 4. Employing one of the solutions to the quantum Yang-Baxter equation we define a statistical vertex model and show that its partition functions on a square lattice with fixed boundary conditions yield ordinary skew Hall-Littlewood functions.

Section 5. We generalise the discussion of the previous section to periodic boundary conditions on the lattice and show that the associated partition functions can be interpreted as cylindric Hall-Littlewood functions. This section contains in particular the definition of cylindric loops and cylindric skew tableaux adapted to the present discussion of the Verlinde algebra. We will state explicit bijections between periodic lattice configurations and cylindric tableaux. We relate the expansion coefficients of the cylindric Hall-Littlewood functions in terms of monomial symmetric, Schur and ordinary Hall-Littlewood functions to matrix elements of the noncommutative Macdonald functions of Section 3. In particular, we express the inverse Kostka-Foulkes matrix as a noncommutative analogue of a Schur polynomial in the q -boson algebra.

Section 6. We use the second solution of the quantum Yang-Baxter equation in Section 3 to define another statistical vertex model whose partition functions lead to cylindric generalisations of the skew Macdonald functions $P_{\lambda/\mu}(t, 0)$. Similar as in the Hall-Littlewood case of the previous section, we relate also their expansion coefficients in various bases in the ring of symmetric functions to matrix elements of noncommutative Macdonald polynomials in the q -boson algebra. We show that the celebrated Kostka-Foulkes polynomials coincide with the matrix element of such a polynomial which is dual to the Schur polynomial.

Section 7. Using the algebraic Bethe ansatz we compute the eigenbasis of the noncommutative Macdonald polynomials in the Fock space. This leads to a set of polynomial equations which define a discrete algebraic variety. We discuss the related coordinate ring and, using invariance under the extended affine symmetric group, identify it as a quotient of the spherical Hecke algebra, which is closely related to the Ringel-Hall algebra of the Jordan quiver. Furthermore it is equipped with the structure of a

commutative Frobenius algebra and we show that its product and coproduct are related to cylindric Hall-Littlewood and Macdonald functions discussed in Sections 5 and 6. Choosing a distinguished basis its structure constants are polynomials in an indeterminate $t = q^2$ whose constant terms equal the WZNW fusion coefficients, the structure constants of the Verlinde algebra.

Section 8. We summarise our findings and set them into relation with a recent observation in the context of four-dimensional $N = 2$ supersymmetric gauge theories which suggests a correspondence between two-dimensional topological quantum field theories and integrable quantum many-body systems.

2 Preliminaries

2.1 q -numbers

Let q be an indeterminate then we define the following standard q -numbers,

$$[m]_q := \frac{q^m - q^{-m}}{q - q^{-1}}, \quad [m]_q! := [m]_q [m-1]_q \cdots [2]_q [1]_q.$$

In addition, we require the q -Pochhammer symbol

$$(x; q)_\infty := \prod_{r=0}^{\infty} (1 - xq^r), \quad (x; q)_r := \prod_{s=0}^{r-1} (1 - xq^s) = \frac{(x; q)_\infty}{(xq^r; q)_\infty}. \quad (2.1)$$

For any composition $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathbb{Z}_{\geq 0}^r$ we will use the shorthand notations

$$(x; q)_\lambda := \prod_{i>0} (x; q)_{\lambda_i}, \quad (q)_r := (q; q)_r, \quad (q)_\lambda := (q)_{\lambda_1} \cdots (q)_{\lambda_r}. \quad (2.2)$$

We will also need the q -binomial coefficients

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_q := \begin{cases} \frac{(q)_{m+n}}{(q)_m (q)_n} = \frac{(1-q^{m+n}) \cdots (1-q^{n+1})}{(1-q) \cdots (1-q^n)}, & m, n \geq 0 \\ 0, & \text{else} \end{cases} \quad (2.3)$$

N.B. q plays here the role of a dummy variable and we will apply the same definitions for other indeterminates or powers of q in particular we will often use $t = q^2$ instead of q .

2.2 The extended affine symmetric group

We recall the definition of the extended affine symmetric group $\hat{\mathfrak{S}}_r$. The latter is generated by the elements $\{\sigma_0, \sigma_1, \dots, \sigma_{r-1}, \tau^{\pm 1}\}$ subject to the relations

$$\begin{aligned} \sigma_i^2 &= 1, & \tau \sigma_i \tau^{-1} &= \sigma_{i+1} \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, & \sigma_i \sigma_j &= \sigma_j \sigma_i, \quad |i-j| > 1, \quad i, j \in \mathbb{Z}_r. \end{aligned} \quad (2.4)$$

Each $w \in \hat{\mathfrak{S}}_r$ can be written as $w = \tau^m \sigma$ for some $m \in \mathbb{Z}$ and $\sigma \in \mathfrak{S}_r$ with $\mathfrak{S}_r \subset \hat{\mathfrak{S}}_r$ being the symmetric group on r -letters generated by $\{\sigma_1, \dots, \sigma_{r-1}\}$. The Bruhat order can be extended from \mathfrak{S}_r to $\hat{\mathfrak{S}}_r$ by setting $w < w'$ if $w = \tau^m \sigma, w' = \tau^{m'} \sigma'$ with $m = m'$ and $\sigma < \sigma'$. Similarly, one can use the decomposition $w = \tau^m \sigma$ to define the length function as $\ell(w) = \ell(\sigma)$. We shall denote the longest element in \mathfrak{S}_r by w_r .

Note that an alternative set of generators for $\hat{\mathfrak{S}}_r$ is $\{\sigma_1, \dots, \sigma_{r-1}\} \cup \{y_1, \dots, y_r\}$ where $y_i y_j = y_j y_i$ for all $1 \leq i, j \leq r$ and $\sigma_i y_i \sigma_i = y_{i+1}$, $\sigma_i y_j = y_j \sigma_i$ for $j \neq i, i+1$. Both definitions are related via $\sigma_0 = \sigma_{r-1} \cdots \sigma_2 \sigma_1 \sigma_2 \cdots \sigma_{r-1} y_1^{-1} y_r$ and $\tau = \sigma_1 \sigma_2 \cdots \sigma_{r-1} y_r$.

2.3 Action on the weight lattice

Let $\mathcal{P}_r = \bigoplus_{i=1}^r \mathbb{Z} \epsilon_i$ be the $\mathfrak{gl}(r)$ weight lattice with standard basis $\epsilon_1, \dots, \epsilon_r$ and inner product $(\epsilon_i, \epsilon_j) = \delta_{i,j}$. We denote the simple roots by $\alpha_i = \epsilon_i - \epsilon_{i+1}$, $i = 1, \dots, r-1$ and the affine root by $\alpha_r = \epsilon_r - \epsilon_1$. Let \mathcal{P}_r^\pm denote the set of (integral) *dominant* and *anti-dominant* weights. Recall the following right level s action of $\hat{\mathfrak{S}}_r$ on \mathcal{P}_r for $s \geq 1$:

$$\begin{aligned} \lambda \sigma_i &= (\lambda_1, \dots, \lambda_{i+1}, \lambda_i, \dots, \lambda_r), \quad i = 1, \dots, r-1, \\ \lambda \sigma_0 &= (\lambda_r + s, \lambda_2, \dots, \lambda_{r-1}, \lambda_1 - s), \\ \lambda \tau &= (\lambda_r + s, \lambda_1, \lambda_2, \dots, \lambda_{r-1}), \\ \lambda y_i &= (\lambda_1, \dots, \lambda_i + s, \dots, \lambda_r). \end{aligned} \tag{2.5}$$

The subsets

$$\mathcal{A}_{r,s}^+ := \{(\lambda_1, \lambda_2, \dots, \lambda_r) \in \mathcal{P}_r^+ \mid s \geq \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 1\} \tag{2.6}$$

and $\mathcal{A}_{r,s}^- = w_r \mathcal{A}_{r,s}^+$ are both fundamental domains with respect to the level s action of $\hat{\mathfrak{S}}_r$ on \mathcal{P}_r . For each $\lambda \in \mathcal{A}_{r,s}^-$ denote by $\mathfrak{S}_\lambda \subset \mathfrak{S}_r$ the stabilizer subgroup of λ and by \mathfrak{S}^λ the set of minimal length representatives of the cosets $\mathfrak{S}_\lambda \backslash \hat{\mathfrak{S}}_r$. We shall use the symbol w_λ for the longest element in \mathfrak{S}_λ .

Throughout this article we will make use of the following bijections.

Reduction. For practical reasons we will also need the set

$$\tilde{\mathcal{A}}_{r,s}^+ := \{(\lambda_1, \lambda_2, \dots, \lambda_r) \in \mathcal{P}_r^+ \mid s > \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 0\}$$

which again is a fundamental domain. We note that there exists a bijection $\sim : \mathcal{A}_{r,s}^+ \rightarrow \tilde{\mathcal{A}}_{r,s}^+$ by sending λ to $\tilde{\lambda}$, the partition obtained from λ by deleting all parts of size s . We shall make frequently use of this map.

***-involution.** In addition, we will require the following *-involution on $\mathcal{A}_{r,s}^+$: given $\lambda \in \mathcal{A}_{r,s}^+$ define λ^* to be the unique element which is the inverse image of $(s - \lambda_r, \dots, s - \lambda_2, s - \lambda_1) \in \tilde{\mathcal{A}}_{r,s}^+$ under the above bijection $\sim : \mathcal{A}_{r,s}^+ \rightarrow \tilde{\mathcal{A}}_{r,s}^+$. Note that when identifying partitions with weights, λ^* is simply the contragredient weight of λ .

Rotation. The $\widehat{\mathfrak{gl}}(n)$ Dynkin diagram automorphism induces a bijection $\text{rot} : \mathcal{A}_{k,n}^+ \rightarrow \mathcal{A}_{k,n}^+$ given by $\lambda \mapsto \text{rot}(\lambda) := \mu$ with $m_i(\mu) := m_{i+1}(\lambda)$, $i \in \mathbb{Z}_n$.

2.4 The affine Hecke algebra

The affine Hecke algebra \hat{H}_r is the $\mathbb{C}[q, q^{-1}]$ algebra generated by $\{T_0, T_1, \dots, T_{r-1}\}$ and an invertible element $T_\tau \equiv \tau$ subject to the relations

$$(T_i - q^{-1})(T_i + q) = 0, \quad \tau T_i \tau^{-1} = T_{i+1}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i, \quad |i - j| > 1, \quad i, j \in \mathbb{Z}_r$$

A basis $\{T_w\}_{w \in \hat{\mathfrak{S}}_r}$ is constructed as follows: for any $w, w' \in \hat{\mathfrak{S}}_r$ set $T_{ww'} := T_w T_{w'}$ if $\ell(w) + \ell(w') = \ell(ww')$ with $T_{\sigma_i} \equiv T_i$. Alternatively, \hat{H}_r is in the *Bernstein presentation* generated by $\{T_1, \dots, T_{r-1}\}$ and a set of commuting, invertible elements $\{Y_1, \dots, Y_r\}$ obeying

$$T_i Y_i T_i = Y_{i+1}, \quad T_i Y_j = Y_j T_i \quad \text{for } j \neq i, i+1.$$

To relate this with the previous presentation use the formulae $T_0 = T_{r-1}^{-1} \dots T_2^{-1} T_1^{-1} T_2^{-1} \dots T_{r-1}^{-1} Y_1^{-1} Y_r$ and $\tau = T_1^{-1} T_2^{-1} \dots T_{r-1}^{-1} Y_r$. Conversely, $Y_i = T_{i-1} T_{i-2} \dots T_1 T_0^{-1} T_{r-1}^{-1} T_{r-2}^{-1} \dots T_i$ and $Y^\lambda = T_{y^\lambda}^{-1}$ for $\lambda \in \mathcal{P}_r^+$. There exists a canonical bar involution $- : \hat{H}_r \rightarrow \hat{H}_r$ defined by $\bar{q} = q^{-1}$ and $\bar{T}_w = (T_{w^{-1}})^{-1}$; in particular $\bar{T}_i = T_i - (q - q^{-1})$.

Define $\mathbf{1}_r := (1/c_r) \sum_{w \in \mathfrak{S}_r} q^{-\ell(w)} T_w$ with $c_r = \sum_w q^{-\ell(w)}$ and denote by $\mathcal{Z}(\hat{H}_r) \cong \mathbb{C}[q, q^{-1}][Y_1, \dots, Y_r]^{\mathfrak{S}_r}$ the centre of the affine Hecke algebra. Then the map $\Phi : \mathcal{Z}(\hat{H}_r) \rightarrow \mathbf{1}_r \hat{H}_r \mathbf{1}_r$ given by $\frac{c_\lambda}{c_r} P_\lambda(Y_1, \dots, Y_r; 0, t = q^2) \mapsto \mathbf{1}_r Y^\lambda \mathbf{1}_r$, where $P_\lambda(0, t)$ are the Hall-Littlewood polynomials (see below), is known as the *Satake isomorphism* and $\mathbf{1}_r \hat{H}_r \mathbf{1}_r$ as the *spherical Hecke algebra*. Recall that $\{\mathbf{1}_r Y^\lambda \mathbf{1}_r : \lambda_1 \geq \dots \geq \lambda_r \geq 0\}$ is a basis of the spherical Hecke algebra; see e.g. [58] for details and references.

2.5 The quantum enveloping algebra of $\widehat{\mathfrak{gl}}(n)$

Let $\hat{U}_n = U'_q \widehat{\mathfrak{gl}}(n)$ be the unital associative $\mathbb{C}(q)$ -algebra generated by $\{E_i, F_i, K_i^{\pm 1}\}_{i=1, \dots, n}$ and subject to the relations:

(R1) The $K_i^{\pm 1}$'s commute with one another and $K_i K_i^{-1} = K_i^{-1} K_i = 1$.

(R2) $K_i E_j = q^{\delta_{ij} - \delta_{ij+1}} E_j K_i$, $K_i F_j = q^{-\delta_{ij} + \delta_{ij+1}} F_j K_i$ and

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_{i,i+1} - K_{i+1,i}}{q - q^{-1}}$$

(R3) For $X = E, F$ we have $X_i X_j = X_j X_i$ for $|i - j| > 1$ and else

$$\begin{aligned} X_i^2 X_{i+1} - (q + q^{-1}) X_i X_{i+1} X_i + X_{i+1} X_i^2 &= 0, \\ X_i X_{i+1}^2 - (q + q^{-1}) X_{i+1} X_i X_{i+1} + X_{i+1}^2 X_i &= 0. \end{aligned} \tag{2.7}$$

Here $K_{i,j} := K_i K_j^{-1}$ and all indices are understood modulo n .

We will denote by $\hat{U}_n = U'_q \widehat{\mathfrak{sl}}(n)$ the subalgebra generated by E_i, F_i and $K_{i,i+1}, K_{i+1,i}$ and by U_n, \mathbf{U}_n the finite-dimensional subalgebras obtained when restricting the index to $i = 1, \dots, n-1$. We choose to work with the coproduct defined via $\Delta(K_i) = K_i \otimes K_i$ and

$$\Delta(E_i) = E_i \otimes K_{i+1,i} + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_{i,i+1} \otimes F_i. \tag{2.8}$$

This will allow us to make contact with the discussion in [9, Section 3, page 7]. The corresponding co-unit and antipode are respectively given by

$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i^{\pm 1}) = 1 \quad (2.9)$$

and

$$S(E_i) = -E_i K_{i,i+1}, \quad S(F_i) = -K_{i+1,i} F_i, \quad S(K_i^{\pm 1}) = K_i^{\mp 1}. \quad (2.10)$$

Furthermore, for discussing the canonical basis below we will require the *bar involution*; this is the unique antilinear automorphism defined via $\bar{E}_i = E_i$, $\bar{F}_i = F_i$ and $\bar{K}_i = K_i^{-1}$. A U_n -module V is said to possess a compatible bar involution $V \rightarrow V$ if $\overline{uv} = \bar{u}\bar{v}$ for all $u \in U_n$ and $v \in V$.

2.6 Macdonald functions

Let q, t be indeterminates and consider the following extension of the ring of symmetric functions $\Lambda(q, t) = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}(q, t)$ where $\Lambda = \varprojlim \Lambda_n$ is the projective limit of the projective system of symmetric polynomials in n variables, $\Lambda_n = \mathbb{Z}[x_1, \dots, x_n]^{\mathfrak{S}_n}$, $n \geq 1$, with the canonical map $\Lambda_{n+1} \twoheadrightarrow \Lambda_n$ which sends x_{n+1} to zero. We now review the definition of a special basis in $\Lambda(q, t)$, known as *symmetric Macdonald functions*; we shall follow the conventions used in [53].

For a given partition λ define

$$b_\lambda(q, t) = \prod_{s \in \lambda} b_\lambda(s; q, t), \quad b_\lambda(s; q, t) := \frac{1 - q^{a_\lambda(s)} t^{l_\lambda(s)+1}}{1 - q^{a_\lambda(s)+1} t^{l_\lambda(s)}}, \quad (2.11)$$

where the product runs over all squares $s = (i, j)$ in the Young diagram of λ and $a_\lambda(s) = \lambda_i - j$ is the arm-length (number of squares to the east) and $l_\lambda(s) = \lambda'_j - i$ the leg-length (number of squares to the south). Given a skew diagram λ/μ , define two functions $\varphi_{\lambda/\mu}, \psi_{\lambda/\mu}$ which are zero unless $\lambda - \mu$ is a horizontal r -strip in which case

$$\varphi_{\lambda/\mu}(q, t) = \prod_{s \in C_{\lambda/\mu}} \frac{b_\lambda(s; q, t)}{b_\mu(s; q, t)} \quad \text{and} \quad \psi_{\lambda/\mu}(q, t) = \prod_{s \in R_{\lambda/\mu} - C_{\lambda/\mu}} \frac{b_\mu(s; q, t)}{b_\lambda(s; q, t)} \quad (2.12)$$

with $C_{\lambda/\mu}$ (respectively $R_{\lambda/\mu}$) being the union of all columns (respectively rows) which intersect λ/μ .

Let λ, μ be partitions with $\mu \subset \lambda$ and denote by λ/μ the associated skew diagram. Given a (semi-standard¹) tableau T of shape λ/μ decompose it into a sequence of partitions $\mu = \lambda^{(0)} \subset \lambda^{(1)} \subset \dots \subset \lambda^{(r)} = \lambda$ such that $\lambda^{(i+1)}/\lambda^{(i)}$ is a horizontal strip and set $\varphi_T := \prod_{i \geq 0} \varphi_{\lambda^{(i+1)}/\lambda^{(i)}}$, $\psi_T := \prod_{i \geq 0} \psi_{\lambda^{(i+1)}/\lambda^{(i)}}$. N.B. we have the identity $b_\mu \varphi_T = b_\lambda \psi_T$ for any tableau T .

Example 2.1 Let $\lambda = (3, 2, 2, 1)$, $\mu = \emptyset$, the empty partition, and consider all tableaux of weight $(2, 2, 2, 2)$,

$$\begin{array}{|c|c|c|} \hline 1 & 1 & 2 \\ \hline 2 & 3 & \\ \hline 3 & 4 & \\ \hline 4 & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 1 & 3 \\ \hline 2 & 2 & \\ \hline 3 & 4 & \\ \hline 4 & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline 1 & 1 & 4 & \\ \hline 2 & 2 & & \\ \hline 3 & 3 & & \\ \hline 4 & & & \\ \hline \end{array} \quad .$$

¹In this article we will only consider semi-standard tableaux and, henceforth, simply call them “tableaux”.

Then one finds the weights

$$\psi_T = \frac{(1+q)(1-t)}{1-qt}, \quad \frac{(1-q^2t)(1-t^2)}{(1-qt)(1-qt^2)}, \quad \frac{(1-q^2t^2)(1-t^3)}{(1-qt^2)(1-qt^3)},$$

and $\varphi_T = b_\lambda \psi_T$ with

$$b_\lambda(q, t) = \frac{(1-t)^3(1-t^2)(1-qt^2)(1-qt^3)^2(1-q^2t^4)}{(1-t)^3(1-t^2)(1-qt^2)(1-qt^3)^2(1-q^2t^4)}.$$

Skew Macdonald functions $Q_{\lambda/\mu}(x; q, t) = b_\lambda(q, t)P_\lambda(x; q, t)/b_\mu(q, t)$ can be defined as the following weighted sums over semi-standard tableaux T ,

$$Q_{\lambda/\mu}(x; q, t) = \sum_{|T|=\lambda/\mu} \varphi_T(q, t)x^T \quad \text{and} \quad P_{\lambda/\mu}(x; q, t) = \sum_{|T|=\lambda/\mu} \psi_T(q, t)x^T. \quad (2.13)$$

Specialising to $\mu = \emptyset$, the empty partition, one obtains ordinary, non-skew, Macdonald functions which are simply denoted by Q_λ, P_λ .

Theorem 2.1 (Macdonald) *The family $\{Q_\lambda(q, t) : \lambda \text{ partition}\}$ forms a basis of the ring of symmetric functions $\Lambda(q, t)$.*

Fix a bilinear form $\Lambda(q, t) \times \Lambda(q, t) \rightarrow \mathbb{C}(q, t)$ (antilinear in the first factor) by setting

$$(P_\lambda, Q_\mu) \mapsto \langle P_\lambda, Q_\mu \rangle_{q, t} := \delta_{\lambda\mu}. \quad (2.14)$$

Macdonald functions interpolate between various other bases in the ring of symmetric functions and we shall make repeated use of the following special cases.

2.6.1 Special cases of Macdonald functions

Elementary symmetric functions. Suppose $\lambda = (1^r)$ is a vertical strip, then $P_{(1^r)} = e_r$ with

$$E(u) = \prod_{i \geq 0} (1 + ux_i) = \sum_{r \geq 0} e_r u^r, \quad e_r = \sum_{i_1 < \dots < i_r} x_{i_1} \cdots x_{i_r}. \quad (2.15)$$

The set $\{e_\lambda\}$ where λ ranges over the partitions and $e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots$ is a \mathbb{Z} -basis of the ring of symmetric functions Λ .

Basic Macdonald functions. Suppose $\lambda = (r)$ is a horizontal r -strip, then $Q_{(r)} = g_r$ with

$$G(u) = \prod_{i \geq 1} \frac{(t x_i u; q)_\infty}{(x_i u; q)_\infty} = \sum_{r \geq 0} g_r(x; q, t) u^r, \quad g_r(q, t) = \sum_{|\mu|=r} \frac{(t; q)_\mu}{(q; q)_\mu} m_\mu, \quad (2.16)$$

where $m_\mu(x) = P_\lambda(x; q, 1)$ are the *monomial symmetric functions*. The set $\{g_\lambda(q, t)\}$ where λ ranges over the partitions and $g_\lambda(q, t) := g_{\lambda_1}(q, t) g_{\lambda_2}(q, t) \cdots$ is a basis in $\Lambda(q, t)$. Below we will consider the special limits $g_\lambda := g_\lambda(0, t)$ and $g'_\lambda := g_\lambda(q, 0)$. In particular, we have that

$$g'_r(q) := g_r(q, 0) = \sum_{\mu \vdash r} \frac{m_\mu}{(q)_\mu}, \quad (q)_\mu := (q)_{\mu_1} (q)_{\mu_2} \cdots \quad (2.17)$$

are multivariate Rogers-Szegö polynomials [62] $h'_r = (q)_r g'_r$ with generating function [1, Chapter 3, Example 17]

$$G'(u; q) = \prod_{i \geq 0} \frac{1}{(ux_i; q)_\infty} = \sum_{r \geq 0} h'_r \frac{u^r}{(q)_r}. \quad (2.18)$$

Hall-Littlewood functions. Specialising $q = 0$ the Macdonald functions become *Hall-Littlewood (HL) functions* sometimes also called *spherical Macdonald functions* because of their relation to the spherical Hecke algebra,

$$Q_\lambda(x; 0, t) = \sum_{|T|=\lambda} \varphi_T(0, t) x^T \quad \text{and} \quad P_\lambda(x; 0, t) = \sum_{|T|=\lambda} \psi_T(0, t) x^T, \quad (2.19)$$

where the coefficients $\varphi_{\lambda/\mu}(0, t) =: \varphi_{\lambda/\mu}(t)$, $\psi_{\lambda/\mu}(0, t) =: \psi_{\lambda/\mu}(t)$ now have the simpler form

$$\varphi_{\lambda/\mu}(t) = \begin{cases} \prod_{i \in I_{\lambda/\mu}} (1 - t^{\lambda'_i - \lambda'_{i+1}}), & \text{if } \lambda - \mu \text{ is a horizontal strip} \\ 0, & \text{otherwise} \end{cases} \quad (2.20)$$

and

$$\psi_{\lambda/\mu}(t) = \begin{cases} \prod_{i \in J_{\lambda/\mu}} (1 - t^{\mu'_i - \mu'_{i+1}}), & \text{if } \lambda/\mu \text{ is a horizontal strip} \\ 0, & \text{otherwise} \end{cases}. \quad (2.21)$$

Here the index set $I_{\lambda/\mu}$ contains all integers i for which $\theta'_i = 1$ and $\theta'_{i+1} = 0$ with $\theta' = \lambda' - \mu'$ being the transposed skew-diagram. In contrast, $J_{\lambda/\mu}$ consists of the integers i for which $\theta'_i = 0$ and $\theta'_{i+1} = 1$. The integers $m_i(\lambda) = \lambda'_i - \lambda'_{i+1}$ and $m_i(\mu) = \mu'_i - \mu'_{i+1}$ are the multiplicities of the part i in the partitions λ and μ , respectively. Note that

$$b_\mu(t) \varphi_{\lambda/\mu}(t) = b_\lambda(t) \psi_{\lambda/\mu}(t), \quad b_\lambda(t) = \prod_{i \geq 0} (t)_{m_i(\lambda)}. \quad (2.22)$$

In what follows we will omit the dependence on the second parameter in the notation and simply write $Q_\lambda(x; t) = Q_\lambda(x; 0, t)$ and $P_\lambda(x; t) = P_\lambda(x; 0, t)$. Denote by R_{ij} the familiar raising and lowering operators of the ring of symmetric functions, $R_{ij}\lambda = (\lambda_1, \dots, \lambda_i + 1, \dots, \lambda_j - 1, \dots)$. Then we have the expression

$$Q_\lambda = \prod_{i < j} \frac{1 - R_{ij}}{1 - t R_{ij}} g_\lambda, \quad g_\lambda = g_{\lambda_1} g_{\lambda_2} \dots, \quad (2.23)$$

which expresses the Hall-Littlewood Q -function as polynomial in the $g_r = g_r(0, t)$.

Demazure Characters and q -Whittaker functions. In light of the definition (2.19) it is natural to consider also the complementary limit of Macdonald polynomials setting now $t = 0$. Define $\psi'_{\lambda/\mu}(q) := \psi_{\lambda/\mu}(q, 0)$ then

$$\psi'_{\lambda/\mu}(q) = \begin{cases} \prod_{i \geq 0} \left[\begin{smallmatrix} \lambda_i - \lambda_{i+1} \\ \lambda_i - \mu_i \end{smallmatrix} \right]_q, & \text{if } \lambda/\mu \text{ is a horizontal strip} \\ 0, & \text{otherwise} \end{cases} \quad (2.24)$$

and, similarly, setting $\varphi'_{\lambda/\mu}(q) := \varphi_{\lambda/\mu}(q, 0)$ one finds

$$\varphi'_{\lambda/\mu}(q) = \begin{cases} \frac{1}{(q; q)_{\lambda_1 - \mu_1}} \prod_{i \geq 0} \left[\begin{smallmatrix} \mu_i - \mu_{i+1} \\ \lambda_{i+1} - \mu_{i+1} \end{smallmatrix} \right]_q, & \text{if } \lambda/\mu \text{ is a horizontal strip} \\ 0, & \text{otherwise} \end{cases}. \quad (2.25)$$

N.B. the identities $\varphi'_{\lambda/\mu}(q) = \frac{b_\lambda(q,0)}{b_\mu(q,0)} \psi_{\lambda/\mu}(q,0) = \frac{b_{\mu'}(0,q)}{b_{\lambda'}(0,q)} \psi'_{\lambda/\mu}(q)$ hold with μ', λ' denoting the conjugate partitions of λ, μ . We shall denote the resulting skew functions by

$$P'_{\lambda/\mu}(x; q) = \sum_{|T|=\lambda/\mu} \psi'_T(q) x^T \quad \text{and} \quad Q'_{\lambda/\mu}(x; q) = \sum_{|T|=\lambda/\mu} \varphi'_T(q) x^T. \quad (2.26)$$

As mentioned in the introduction for $\mu = \emptyset$ these specialisations of Macdonald functions coincide with certain Demazure characters [63]. In [26] they have been named *q-deformed* or simply *q-Whittaker functions*. For general μ these links have not been established and, therefore, I shall refer to them simply as skew Macdonald functions, although it will always be understood that $t = 0$. Applying the involution $\omega : \Lambda \rightarrow \Lambda$ defined via $s_\lambda \mapsto s_{\lambda'}$, where $s_\lambda = \det(e_{\lambda'_i - i + j})$ is the Schur function, we obtain from the Macdonald functions P'_λ, Q'_λ the so-called *modified Hall-Littlewood functions*. The latter form dual bases of the ordinary HL functions with respect to the standard inner product $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$. That is, we have $\tilde{Q}_{\lambda'} = \omega P'_\lambda$ and $\tilde{P}_{\lambda'} = \omega Q'_\lambda$ with $\langle \tilde{Q}_\lambda, P_\mu \rangle = \langle \tilde{P}_\lambda, Q_\mu \rangle = \delta_{\lambda\mu}$.

2.6.2 Hall algebra, coproduct and skew Hall-Littlewood functions

One of the reasons for the prominence of Hall-Littlewood functions is their connection with the Hall algebra. When the variable t is evaluated as the cardinality of a finite field it is well-known that the coefficients in the product expansion (1.3) of Hall-Littlewood P -functions are related to Hall polynomials, the structure constants of Steinitz's Hall algebra. For generic t the expansion coefficients $f'_{\lambda\mu}(t)$ are polynomials which vanish identically unless the Littlewood-Richardson coefficient $f'_{\lambda\mu}(0) = c'_{\lambda'\mu'} = c'_{\lambda\mu}$ is nonzero.

Skew Hall-Littlewood functions are intimately linked to the product expansion (1.3) through the following construction: endow $\Lambda(t) = \Lambda(0, t)$ with the coproduct $\Delta : \Lambda(t) \rightarrow \Lambda(t) \otimes \Lambda(t)$ which is the projective limit of the natural embedding $\mathbb{C}[x_1, \dots, x_{2n}]^{\mathfrak{S}_{2n}} \hookrightarrow \mathbb{C}[x_1, \dots, x_n]^{\mathfrak{S}_n} \otimes \mathbb{C}[x_{n+1}, \dots, x_{2n}]^{\mathfrak{S}_n}$. The specialisation of the bilinear form (2.14) at $q = 0$ yields the unique inner product $\langle \cdot, \cdot \rangle_t$ on $\Lambda(t)$ such that

$$\langle f, gh \rangle_t = \langle \Delta f, g \otimes h \rangle_t \quad \text{and} \quad \langle p_m, p_n \rangle_t = \delta_{m,n} \frac{m}{t^m - 1}, \quad (2.27)$$

where $p_m = \sum_i x_i^m$ is the m^{th} power sum, the latter generate the \mathbb{Q} -algebra $\Lambda^{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ of symmetric functions. The co-algebra of symmetric functions $(\Lambda(t), \Delta)$ can be turned into a bi-algebra with respect to the co-unit $\varepsilon(f) := f(0, 0, \dots)$ and the first inner product identity in (2.27) ensures that this bi-algebra is self-dual. In particular, one has the identities

$$\Delta Q_\lambda = \sum_{\mu} Q_{\lambda/\mu} \otimes Q_\mu \quad \text{and} \quad Q_{\lambda/\mu} = \sum_{\nu} f_{\mu\nu}^\lambda Q_\nu, \quad (2.28)$$

where $\langle Q_{\lambda/\mu}, P_\nu \rangle_t = \langle Q_\lambda, P_\mu P_\nu \rangle_t = f_{\mu\nu}^\lambda(t)$ are the coefficients in (1.3).

Using the well-known duality relation of Macdonald functions [53, VI.5, Equation (5.1), p327] the analogous Hopf algebra structure can be defined on the Macdonald P', Q' -functions: set $q = t$ and define an automorphism $\omega_t : \Lambda(t, 0) \rightarrow \Lambda(0, t)$ through the following table

F	Q'_λ	P'_λ	e_λ	g'_λ	s_λ	S'_λ
$\omega_t F$	$P_{\lambda'}$	$Q_{\lambda'}$	g_λ	e_λ	$S_{\lambda'}$	$s_{\lambda'}$

(2.29)

where $F'_\lambda := F_\lambda(t, 0)$ and $F_{\lambda'} := F_{\lambda'}(0, t)$ for $F = Q, P, g$. Each single column in the table fixes ω_t uniquely as all the displayed functions are bases in $\Lambda(t, 0)$ and $\Lambda(0, t)$ respectively. Here we have introduced the dual functions

$$S_\lambda = \det(g_{\lambda_i - i + j}) \quad \text{and} \quad S'_\lambda = \det(g'_{\lambda_i - i + j}) . \quad (2.30)$$

of the Schur function with respect to the two inner products obtained from (2.14), $\langle S_\lambda, s_\mu \rangle_{0,t} = \langle s_{\lambda'}, S'_{\mu'} \rangle_{q,0} = \delta_{\lambda\mu}$.

Theorem 2.2 (Macdonald) *The map $\omega_t : \Lambda(t, 0) \rightarrow \Lambda(0, t)$ defined via $Q_\lambda(t, 0) \mapsto P_{\lambda'}(0, t)$ (resp. $P_\lambda(t, 0) \mapsto Q_{\lambda'}(0, t)$) is a $\mathbb{C}(t)$ -bialgebra isomorphism which preserves the inner product (2.27). Thus, in particular, we have that*

$$Q'_\lambda Q'_\mu = \sum_\nu f_{\lambda'\mu'}^{\nu'}(t) Q'_\nu \quad \text{and} \quad P'_{\lambda/\mu} = \sum_\nu f_{\mu'\nu'}^{\lambda'}(t) P'_\nu , \quad (2.31)$$

where λ', μ', ν' are the conjugate partitions of λ, μ, ν .

2.6.3 Generalised Cauchy identities

We recall the following well known generalisation of Cauchy's identity to Hall-Littlewood functions [53, III.4],

$$\begin{aligned} \prod_{i,j} \frac{1 - tx_i y_j}{1 - x_i y_j} &= \sum_\lambda g_\lambda(x; t) m_\lambda(y) \\ &= \sum_\lambda s_\lambda(x) S_\lambda(y; t) = \sum_\lambda Q_\lambda(x; t) P_\lambda(y; t) . \end{aligned}$$

Applying the inverse of the automorphism ω_t to the functions in the x variables once we obtain

$$\begin{aligned} \prod_{i,j} (1 + x_i y_j) &= \sum_\lambda e_\lambda(x) m_\lambda(y) \\ &= \sum_\lambda s_{\lambda'}(x) s_\lambda(y) = \sum_\lambda P'_{\lambda'}(x; t) P_\lambda(y; t) \end{aligned} \quad (2.32)$$

and doing so for a second time yields

$$\begin{aligned} \prod_{i,j} \frac{1}{(x_i y_j; t)_\infty} &= \sum_\lambda g'_\lambda(x; t) m_\lambda(y) \\ &= \sum_\lambda S'_\lambda(x; t) s_\lambda(y) = \sum_\lambda Q'_{\lambda'}(x; t) P'_{\lambda'}(y; t) \end{aligned} \quad (2.33)$$

where in order to arrive at the last relation we have swapped x and y -variables.

3 q -bosons and Yang-Baxter algebras

In this section we introduce the basic noncommutative algebraic structure, the q -oscillator or boson algebra, from which we will construct step-by-step all the other relevant algebraic objects in the following order: a q -Schur algebra of \hat{U}_n , solutions to the Yang-Baxter equation and their associated Yang-Baxter algebras as well as noncommutative analogues of symmetric polynomials.

3.1 q-boson algebra

There exist different versions of the q -boson algebra, also called the q -oscillator or Heisenberg algebra in the literature; see for instance Chapter 5 in [38] as well as references therein. We shall work with the following version (we assume henceforth that $q^{\pm 1}$ exist); compare with the symmetric q -oscillator algebra in [38, 5.1.1, Definition 2 and 5.1.2].

Definition 3.1 (q -deformed boson algebra) *The q -boson algebra \mathcal{H}_q is the unital, associative $\mathbb{C}(q)$ -algebra defined in terms of the generators $\{q^{\pm N}, \beta, \beta^*\}$ and the algebraic relations,*

$$q^N q^{-N} = q^{-N} q^N = 1, \quad q^N \beta = \beta q^{N-1}, \quad q^N \beta^* = \beta^* q^{N+1}, \quad (3.1)$$

$$\beta \beta^* - \beta^* \beta = (1 - q^2) q^{2N}, \quad \beta \beta^* - q^2 \beta^* \beta = 1 - q^2, \quad (3.2)$$

where $q^{\pm N}$ denote generators and q^{pN+x} is shorthand for $(q^{\pm N})^p q^x$.

Note that (3.2) implies the relations

$$\beta^* \beta = 1 - q^{2N} \quad \text{and} \quad \beta \beta^* = 1 - q^{2(N+1)}. \quad (3.3)$$

The proof of the following proposition is contained in [38, 5.1.1, Proposition 1].

Proposition 3.1 (basis of the q -boson algebra) *The set $\{(\beta^*)^p q^{rN}, q^{rN} \beta^s : p, s \in \mathbb{N}, r \in \mathbb{Z}, (r, s) \neq (-1, 0)\}$ forms a basis of \mathcal{H}_q .*

In what follows we will consider the n -fold tensor product of the q -oscillator algebra and denote by $\beta_i, \beta_i^*, q^{\pm N_i}$, $i = 1, 2, \dots, n$ the generators which belong to the i^{th} factor of the tensor product $\mathcal{H}_q^{\otimes n}$. The set of generators $\{\beta_i, \beta_i^*, q^{\pm N_i}\}_{i=1}^n$ then obeys the following relations:

$$\beta_i \beta_j - \beta_j \beta_i = \beta_i^* \beta_j^* - \beta_j^* \beta_i^* = q^{N_i} q^{N_j} - q^{N_j} q^{N_i} = 0 \quad (3.4)$$

$$q^{N_i} \beta_j = \beta_j q^{N_i - \delta_{ij}}, \quad q^{N_i} \beta_j^* = \beta_j^* q^{N_i + \delta_{ij}}, \quad (3.5)$$

$$\beta_i \beta_j^* - \beta_j^* \beta_i = \delta_{ij} (1 - q^2) q^{2N_i}, \quad \beta_i \beta_i^* - q^2 \beta_i^* \beta_i = 1 - q^2. \quad (3.6)$$

The following proposition is a generalisation of the case $U_q \mathfrak{sl}(2)$ discussed in [38, 5.1.1, Proposition 3]; compare also with [30] where closely related homomorphisms are discussed for general affine Lie algebras.

Proposition 3.2 (Jordan-Schwinger realisation) *Let $n > 2$ and z be an indeterminate with $\bar{z} = z^{-1}$. There exists a homomorphism $h : U_n \rightarrow \mathcal{H}_q^{\otimes n}$ such that*

$$E_i \mapsto -\frac{q^{-N_i} \beta_i^* \beta_{i+1}}{q - q^{-1}}, \quad F_i \mapsto -\frac{q^{-N_{i+1}} \beta_i \beta_{i+1}^*}{q - q^{-1}}, \quad K_i^{\pm 1} \mapsto q^{\pm N_i} \quad (3.7)$$

where $i = 1, 2, \dots, n-1$. This homomorphism can be extended to the affine algebra $h_z : \hat{U}_n \rightarrow \mathcal{H}_q^{\otimes n} \otimes \mathbb{C}[z, z^{-1}]$ by setting

$$E_n \mapsto -z \frac{q^{-N_n} \beta_n^* \beta_1}{q - q^{-1}}, \quad F_n \mapsto -z^{-1} \frac{q^{-N_1} \beta_n \beta_1^*}{q - q^{-1}}, \quad K_n^{\pm 1} \mapsto q^{\mp N_n}. \quad (3.8)$$

Proof. The proof is a straightforward computation exploiting repeatedly the relations (3.4), (3.5), (3.6) and (3.3). ■

The following infinite-dimensional highest-weight module, the Fock space, is for instance discussed in [38, Chapter 5, Section 5.2] we refer the reader to *loc. cit.* for a proof. We will be using for convenience the bra-ket notation from physics.

Proposition 3.3 (Fock space) *Let $\mathcal{I} \subset \mathcal{H}_q$ be the two-sided ideal generated by β and $q^N - 1$ and set $\mathcal{F} = \mathcal{H}_q/\mathcal{I}$. (i) \mathcal{F} has highest weight vector $|0\rangle = 1 + \mathcal{I}$ and the set $\{|m\rangle := (\beta^*)^m/(q^2)_m|0\rangle \mid m \in \mathbb{Z}_{\geq 0}\}$ forms a basis. The following relations hold,*

$$q^N|m\rangle = q^m|m\rangle, \quad \beta^*|m\rangle = (1 - q^{2m+2})|m+1\rangle, \quad \beta|m\rangle = |m-1\rangle. \quad (3.9)$$

(ii) *The module \mathcal{F} is simple as long as q is not evaluated at a root of unity.*

In what follows we will also make use of the dual basis $\{\langle m| \}_{m \in \mathbb{Z}_{\geq 0}} \subset \tilde{\mathcal{F}}$, i.e. $\langle m|m'\rangle = \delta_{m,m'}$ and

$$\langle m|q^N = q^m\langle m|, \quad \langle m|\beta^* = (1 - q^{2m})\langle m-1|, \quad \langle m|\beta = \langle m+1|. \quad (3.10)$$

It will be sometimes convenient to employ the following vector space isomorphism $\iota : \tilde{\mathcal{F}} \rightarrow \mathcal{F}$ between the Fock space and its dual: map the bra-vector $\langle m|$ onto the ket-vector $(q^2)_m|m\rangle$. This induces a scalar product on \mathcal{F} , which by abuse of notation we also denote by $\langle | \rangle$ and which we assume to be antilinear in the first factor. With respect to this inner product β, β^* are adjoints of each other and $(q^{\pm N})^* = q^{\pm N}$.

Considering the n -fold tensor product $\mathcal{F}^{\otimes n}$ we can parametrize the standard basis $\{|m_1, \dots, m_n\rangle := |m_1\rangle \otimes \dots \otimes |m_n\rangle : m_i \in \mathbb{Z}_{\geq 0}\} \subset \mathcal{F}^{\otimes n}$ in terms of partitions $\lambda \in \mathcal{A}_{k,n}^+$: denote by $\mathcal{F}_k^{\otimes n} \subset \mathcal{F}^{\otimes n}$ the subspace spanned by $\{|\lambda\rangle : \lambda \in \mathcal{A}_{k,n}^+\}$, where $m_i(\lambda)$ is the multiplicity of the part i in λ and $|\lambda\rangle := |m_1(\lambda)\rangle \otimes \dots \otimes |m_n(\lambda)\rangle$. Obviously, we have $\mathcal{F}^{\otimes n} = \bigoplus_{k \geq 0} \mathcal{F}_k^{\otimes n}$ with $\mathcal{F}_0^{\otimes n} = \mathbb{C}(q)|\varnothing\rangle \cong \mathbb{C}(q)$. We denote by $\cup_{k \geq 0} \{|\lambda\rangle : \lambda \in \mathcal{A}_{k,n}^+\}$ the corresponding dual basis with $\langle \lambda|\mu\rangle = \delta_{\lambda,\mu}$. N.B. the above vector space isomorphism $\iota : \tilde{\mathcal{F}} \rightarrow \mathcal{F}$ generalises trivially to $\iota : \tilde{\mathcal{F}}^{\otimes n} \rightarrow \mathcal{F}^{\otimes n}$ by setting $\langle \lambda| \mapsto b_\lambda(q^2)|\lambda\rangle$.

Remark 3.1 *The physical interpretation of the operators β, β^* is that they annihilate and create a q -boson, respectively. The tensor product $\mathcal{F}^{\otimes n}$ then describes highly correlated quantum particles on a one-dimensional lattice with n -sites with m_i being the occupation number at site i .*

Because of the homomorphism (3.7) we can view $\mathcal{F}^{\otimes n}$ as a U_n -module. This module is reducible and there is a natural decomposition into irreducible submodules. To describe the latter we recall some known facts first; compare with [9, Section 3, page 7].

Recall that the vector representation $V = \mathbb{C}\{v_1, \dots, v_n\}$ of U_n associated with the fundamental weight ω_1 is given by

$$E_i v_r = \delta_{i,r-1} v_{r-1}, \quad F_i v_r = \delta_{r,i} v_{r+1}, \quad K_i v_r = q^{\delta_{i,r}} v_r, \quad (3.11)$$

with $i = 1, 2, \dots, n-1$. One easily verifies, that $\bar{v}_r = v_r$ is a compatible bar involution; see the discussion in [9, Section 3] on how to induce compatible bar involutions on tensor products of V .

Define a right action of the Hecke algebra H_k on $V^{\otimes k}$ with $M_\mu.T_j := \mathcal{R}_{j,j+1}^{-1}M_\mu$ where $\{M_\mu := v_{\mu_k} \otimes \cdots \otimes v_{\mu_2} \otimes v_{\mu_1} : 1 \leq \mu_i \leq n\}$ is the standard basis in $V^{\otimes k}$ and $\mathcal{R}^{-1} : V \otimes V \rightarrow V \otimes V$ is given by

$$\mathcal{R}^{-1}(v_r \otimes v_s) = \begin{cases} v_s \otimes v_r, & \text{if } r < s \\ q^{-1}v_r \otimes v_r, & \text{if } r = s \\ v_s \otimes v_r - (q - q^{-1})v_r \otimes v_s, & \text{if } r > s \end{cases} \quad (3.12)$$

for $r, s = 1, \dots, n$ and we have $T_i^2 = (q^{-1} - q)T_i + 1$ as well as $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$. Given a permutation $\sigma \in \mathfrak{S}_k$ set as usual $T_\sigma = T_{i_1} \cdots T_{i_r}$ where $\sigma_{i_1} \cdots \sigma_{i_r}$ is the reduced expression of σ into elementary transpositions. Employing this action of the Hecke algebra we now discuss two different versions of q -analogues of the symmetric tensor algebra which we will then identify with $\mathcal{F}^{\otimes n}$ and its dual $\tilde{\mathcal{F}}^{\otimes n}$.

The k^{th} *divided symmetric power* $S^k(V)$ is defined as $S^k(V) = V^{\otimes k}.X_k$ where

$$X_k = \sum_{w \in \mathfrak{S}_k} q^{\ell(w_k) - \ell(w)} T_w \quad (3.13)$$

is bar-invariant and w_k is the longest element in \mathfrak{S}_k with $\ell(w_k) = k(k-1)/2$. Note that $T_i X_k = X_k T_i = q^{-1} X_k$. Set $S(V) := \bigoplus_{k \in \mathbb{Z}_{\geq 0}} S^k(V)$.

In contrast the *quantum symmetric tensor algebra* $\tilde{S}(V)$ of V is the tensor algebra $T(V) := \bigoplus_{k \geq 0} V^{\otimes k}$ divided by the two sided ideal I generated from the elements $\{v_j \otimes v_i - q^{-1}v_i \otimes v_j : 1 \leq i < j \leq n\}$. Denote by $\tilde{S}^k(V)$ the k -th homogeneous component, that is the invariant subspace under the natural action of the Hecke algebra on $V^{\otimes k}$.

Proposition 3.4 (quantum symmetric tensor algebra) *There exist U_n -module isomorphisms $\mathcal{F}^{\otimes n} \cong S(V)$ and $\tilde{\mathcal{F}}^{\otimes n} \cong \tilde{S}(V)$ such that $\mathcal{F}_k^{\otimes n}$ and $\tilde{\mathcal{F}}_k^{\otimes n}$ are mapped onto $S^k(V)$ and $\tilde{S}^k(V)$, respectively.*

Proof. Following [9, Section 5] define two bases

$$X_\lambda = \frac{1}{[m_1]_q! \cdots [m_n]_q!} M_\lambda X_k \quad \text{and} \quad \tilde{X}_\lambda := \pi_k(M_\lambda) \quad (3.14)$$

in $S^k(V)$ and $\tilde{S}^k(V)$, respectively, where $\lambda \in \mathcal{A}_{k,n}^+$, $m_i = m_i(\lambda)$, $[m]_q := (q^m - q^{-m})/(q - q^{-1})$ and $\pi_k : V^{\otimes k} \twoheadrightarrow \tilde{S}^k(V)$ is the quotient map. We claim that the maps $\mathcal{F}_k^{\otimes n} \ni |\lambda\rangle \mapsto X_\lambda$ and $\tilde{\mathcal{F}}_k^{\otimes n} \ni \langle \lambda| \mapsto \tilde{X}_\lambda$ are U_n -module isomorphisms with U_n acting on $\mathcal{F}_k^{\otimes n}$ via the homomorphism h in (3.7) and on $\tilde{\mathcal{F}}_k^{\otimes n}$ via $h^* \circ \Theta$, where Θ is the algebra anti-automorphism $\Theta(E_i) = F_i$, $\Theta(F_i) = E_i$, $\Theta(K_i) = K_i$ and h^* is the map obtained by taking the adjoint of the image under h .

Exploiting the coproduct (2.8) we compute the following action on a monomial basis vector

$$\begin{aligned} \Delta^k(E_i)M_{\lambda'} &= q^{m_{i+1}-1} M_{(\underbrace{\dots, i, \dots, i, i+1, \dots, i+1, \dots}_{m_i+1} \underbrace{\dots}_{m_{i+1}-1})} \\ &\quad + q^{m_{i+1}-2} M_{(\dots, \underbrace{i, \dots, i, i+1, i, i+1, \dots, i+1, \dots}_{m_i} \underbrace{\dots}_{m_{i+1}-2})} + \cdots \\ &\quad \quad \quad + M_{(\dots, \underbrace{i, \dots, i, i+1, \dots, i+1, \dots}_{m_i} \underbrace{\dots}_{m_{i+1}-1})} \end{aligned}$$

Employing that for any permutation μ of λ we have $M_\mu X_k = q^{-\ell(\mu, \lambda)} X_\lambda$ where $\ell(\mu, \lambda)$ is the length of the shortest permutation which brings μ into λ (see [9, Section 5, eqn (5.3)]) we find that $\Delta^k(E_i)M_\lambda X_k = [m_{i+1}]_q M_\nu X_k$ as well as $\Delta^k(E_i)\pi_k(M_\lambda) = [m_{i+1}]_q \pi_k(M_\nu)$, where $\nu \in \mathcal{A}_{k,n}^+$ is the partition obtained by removing a part $(i+1)$ and adding a part i . Thus, $\Delta^k(E_i)X_\lambda = [m_i + 1]_q X_\nu$ and $\Delta^k(E_i)\tilde{X}_\lambda = [m_{i+1}]_q \tilde{X}_\nu$. In comparison we have according to the homomorphism (3.7) that

$$\frac{\beta_i^* \beta_{i+1} q^{-N_i}}{1 - q^2} |\lambda\rangle = [m_i + 1]_q |\nu\rangle \quad \text{and} \quad \langle \lambda | \frac{\beta_i \beta_{i+1}^* q^{-N_{i+1}}}{1 - q^2} = \langle \nu | [m_{i+1}]_q .$$

The computation for the generator F_i is similar and it is trivial for K_i . ■

Remark 3.2 Let $\iota : S^k(V) \hookrightarrow V^{\otimes k}$ be the inclusion map and $\pi_k : V^{\otimes k} \rightarrow \tilde{S}^k(V)$ the quotient map. There exists a bilinear form $\langle \cdot | \cdot \rangle$ on $V^{\otimes k}$ which induces a pairing $\langle \cdot | \cdot \rangle : \tilde{S}^k(V) \times S^k(V) \rightarrow \mathbb{C}(q)$ by setting $\langle Y | X \rangle := \langle Y | \iota(X) \rangle = \langle \pi_k(Y), X \rangle$ for $X \in S^k(V), Y \in V^{\otimes k}$; see [9, Section 5, para after eqn (5.2)] and references therein. This pairing coincides with bracket of the Fock space and its dual, i.e. $\langle M_\lambda | \iota(X_\mu) \rangle = \langle \pi_k(M_\lambda) | X_\mu \rangle = \langle \lambda | \mu \rangle = \delta_{\lambda\mu}$.

We are now turning to the affine algebra and consider for simplicity $\hat{\mathbf{U}}_n$ instead of \hat{U}_n . First we remind the reader that V can be turned into a so-called evaluation module $V(a) = V \otimes \mathbb{C}[a, a^{-1}]$ for $\hat{\mathbf{U}}_n$ setting

$$E_n v_r = a \delta_{r,1} v_n, \quad F_n v_r = a^{-1} \delta_{r,n} v_1, \quad K_{n,1} v_r = q^{\delta_{r,n} - \delta_{r,1}} v_r . \quad (3.15)$$

We now have the following:

Proposition 3.5 (Kirillov-Reshetikhin module isomorphism) *The vector space $\mathcal{F}_k^{\otimes n} \otimes \mathbb{C}[z, z^{-1}]$ viewed as $\hat{\mathbf{U}}_n$ -module is isomorphic to the Kirillov-Reshetikhin module $W^{1,k} = W(k\omega_1)$. That is, it coincides with the irreducible submodule of highest weight $k\omega_1$ of the following (reducible) tensor product of $\hat{\mathbf{U}}_n$ evaluation modules,*

$$V(zq^{-k+1}) \otimes V(zq^{-k+3}) \otimes \cdots \otimes V(zq^{k-1}) . \quad (3.16)$$

Proof. Employing results in [12–14] on the classification of finite-dimensional type 1 representations of $\hat{\mathbf{U}}_n$ ($\tilde{K} = \tilde{K}_1^{-1} \tilde{K}_2^{-1} \cdots \tilde{K}_{n-1}^{-1}$) and the previous result $\mathcal{F}_k^{\otimes n} \cong S^k(V)$, it suffices to compute the evaluation parameters. Consider in $\mathcal{A}_{k,n}^+$ the partitions $\lambda = n^k = (n, n, \dots, n)$ and $\mu = (n, n, \dots, n, 1)$. Under the previously stated isomorphism of U_n -modules $|\lambda\rangle$ and $|\mu\rangle$ are mapped to the following vectors in $S^k(V)$

$$|\lambda\rangle \mapsto \frac{1}{[k]!} v_n \otimes \cdots \otimes v_n \quad \text{and} \quad |\mu\rangle \mapsto \frac{1}{[k-1]!} \sum_{i=1}^k q^{i-1} v_n \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n .$$

The $\hat{\mathbf{U}}_n$ -action on $\mathcal{F}_k^{\otimes n} \otimes \mathbb{C}[z, z^{-1}]$ yields (compare with (3.8))

$$h_z(E_0)|\mu\rangle = z[k]_q |\lambda\rangle \quad \text{and} \quad h_z(F_0)|\lambda\rangle = z^{-1} |\mu\rangle .$$

In comparison the $\hat{\mathbf{U}}_n$ -action on $V(z_1) \otimes V(z_2) \otimes \cdots \otimes V(z_k)$ gives

$$\begin{aligned} \Delta^k(E_0) v_n \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n &= z_i q^{k-i} v_n \otimes \cdots \otimes v_n, \\ \Delta^k(F_0) v_n \otimes \cdots \otimes v_n &= \sum_{i=1}^k z_i^{-1} q^{i-1} v_n \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n . \end{aligned}$$

Hence, acting with X_k from the right we must have that $z_i = zq^{-k+2i-1}$ as asserted. From this it follows that the Drinfeld polynomials [12–14] are given by $P_1(z) = (1 - zq^{-k+1})(1 - zq^{-k+3}) \cdots (1 - zq^{k-1})$ and $P_l = 1$ for $l \neq 1$ which fixes the Kirillov-Reshetikhin module up to isomorphism. ■

3.2 Solutions to the Yang-Baxter equation

We now discuss three particular solutions to the Yang-Baxter equation in terms of the q -boson algebra. The first one has been previously obtained by Bogoliubov, Izergin and Kitanine [7], the others are new. They are special limits of solutions related to the $U_q \widehat{\mathfrak{sl}}(2)$ R-matrix [41].

Let u be an invertible variable, called the spectral variable. Define [7]

$$L(u) = \begin{pmatrix} 1 & u\beta^* \\ \beta & u1 \end{pmatrix} \in \text{End}[\mathbb{C}^2(u)] \otimes \mathcal{H}_q, \quad (3.17)$$

where the notation is shorthand for $L(u) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes 1 + u \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \beta^* + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \beta + u \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes 1$.

Proposition 3.6 (Bogoliubov, Izergin, Kitanine) *The L -operator satisfies the Yang-Baxter equation*

$$R_{12}(u, v) L_1(u) L_2(v) = L_2(v) L_1(u) R_{12}(u, v), \quad (3.18)$$

where $R \in \text{End}(\mathbb{C}(u)^2 \otimes \mathbb{C}(v)^2) \cong \text{End } \mathbb{C}(u, v)^4$ is given by

$$R(u, v) = \begin{pmatrix} \frac{u-tv}{u-v} & 0 & 0 & 0 \\ 0 & t & \frac{1-t}{u-v}u & 0 \\ 0 & \frac{1-t}{u-v}v & 1 & 0 \\ 0 & 0 & 0 & \frac{u-tv}{u-v} \end{pmatrix} \quad (3.19)$$

with respect to the basis $\{v_0 \otimes v_0, v_0 \otimes v_1, v_1 \otimes v_0, v_1 \otimes v_1\}$ and we have set $t = q^2$.

Proof. Observe that L is a 2×2 matrix with entries

$$\langle \sigma' | L(u) | \sigma \rangle = u^\sigma \frac{\beta^{\sigma'} (\beta^*)^\sigma - t (\beta^*)^\sigma \beta^{\sigma'}}{1-t}, \quad \sigma' \sigma = 0, 1.$$

Making a case-by-case distinction and using repeatedly that $\beta\beta^* - t\beta^*\beta = 1-t$ the proof is a straightforward computation. ■

We now define a second solution to the Yang-Baxter equation, which we then relate to the first, in terms of an “infinite-dimensional matrix”. Setting $\mathcal{F}((u)) = \mathbb{C}((u)) \otimes \mathcal{F}$ let $L'(u) \in \text{End}[\mathcal{F}((u))] \otimes \mathcal{H}_q$ be given by ($t = q^2$)

$$\langle m' | L'(u) | m \rangle = \frac{u^m (\beta^*)^m \beta^{m'}}{(t)_m}, \quad (3.20)$$

where $|m\rangle$ respectively $\langle m'|$ label the basis and dual basis in the Fock space \mathcal{F} of the q -boson algebra \mathcal{H}_q . Define another operator $R' \in \text{End}[\mathcal{F}((u)) \otimes \mathcal{F}((v))]$ via

$$\langle m'_1, m'_2 | R'(u, v) | m_1, m_2 \rangle = \left(\frac{u}{v}\right)^{m_1} \begin{bmatrix} m'_2 \\ m_1 \end{bmatrix}_t (u/v; t)_{m'_2 - m_1} \delta_{m_1 + m_2, m'_1 + m'_2} \quad (3.21)$$

where the q -deformed binomial coefficient is zero if $m'_2 < m_1$, hence only terms survive for which $m'_2 \geq m_1$.

Proposition 3.7 *We have the identity*

$$R'_{12}(u, v) L'_1(u) L'_2(v) = L'_2(v) L'_1(u) R'_{12}(u, v). \quad (3.22)$$

Moreover, the R' -operator is invertible: let $P : \mathcal{F} \otimes \mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{F}$ be the flip operator $P|m_1, m_2\rangle = |m_2, m_1\rangle$, then $R'(u, v) P R'(v, u) P = 1$.

Proof. Exploiting (3.1), (3.2) and (3.3) one first proves via induction the relation

$$\beta^a \tilde{\beta}^b = \sum_{r=0}^{\min(a,b)} t^{(a-r)(b-r)} \begin{bmatrix} b \\ r \end{bmatrix} \frac{[a]!}{[a-r]!} \tilde{\beta}^{b-r} \beta^{a-r}, \quad \tilde{\beta} := \frac{\beta^*}{1-t}$$

with $[x] := (1 - t^x)/(1 - t)$ and then verifies the assertion after a somewhat tedious but straightforward computation whose details we omit. ■

To relate the two solutions, L and L' , we need yet another operator $R'' \in \text{End}[\mathbb{C}(u)^2 \otimes \mathcal{F}(v)]$ defined as

$$R''(u, v) = L(u/v) + \begin{pmatrix} u/v \, q^{2N} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 + u/v \, q^{2N} & u/v \, \beta^* \\ \beta & u/v \cdot 1 \end{pmatrix}. \quad (3.23)$$

Proposition 3.8 *The R'' operator satisfies the identity*

$$R''_{12}(u, v) L_1(u) L'_2(v) = L'_2(v) L_1(u) R''_{12}(u, v) \quad (3.24)$$

and possesses the inverse

$$(R'')^{-1}(u, v) = \frac{1}{1 + u/v} \begin{pmatrix} q^{-2N} & -q^{-2N} \beta^* \\ -v/u \, \beta q^{-2N} & 1 + v/u \, q^{-2N-2} \end{pmatrix}. \quad (3.25)$$

Proof. Once more the claimed identities are a direct consequence of the q -boson algebra relations (3.1), (3.2) and (3.3). ■

3.3 The Yang-Baxter algebra

The Yang-Baxter equation is naturally endowed with a coproduct: one easily verifies that given the solutions L, L' the operators $\Delta L = L_2 L_1 \in \text{End}[\mathbb{C}^2(u)] \otimes \mathcal{H}_q^{\otimes 2}$ and $\Delta L' = L'_2 L'_1 \in \text{End}[\mathcal{F}((u))] \otimes \mathcal{H}_q^{\otimes 2}$ are also solutions to the quantum Yang-Baxter equation. Repeating the argument it is therefore natural to consider the so-called monodromy matrices

$$T(u) = \Delta^n L(u) = L_n(u) L_{n-1}(u) \cdots L_1(u) \in \text{End}[\mathbb{C}^2(u)] \otimes \mathcal{H}_q^{\otimes n} \quad (3.26)$$

and

$$T'(u) = \Delta^n L'(u) = L'_n(u) L'_{n-1}(u) \cdots L'_1(u) \in \text{End}[\mathcal{F}((u))] \otimes \mathcal{H}_q^{\otimes n}. \quad (3.27)$$

Much of the discussion which is to follow will focus on the matrix elements of these two operators. We start the discussion with (3.26).

Rewrite $T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$ and for $\mathcal{O} = A, B, C, D \in \mathbb{C}[u] \otimes \mathcal{H}_q^{\otimes n}$ introduce the series expansions $\mathcal{O}(u) = \sum_{r \geq 0} u^r \mathcal{O}_r$. Note that the latter terminate for $r > n$ according to the definition (3.26) and the L -operator (3.17).

Definition 3.2 The Yang-Baxter algebra $\mathfrak{A} \subset \mathcal{H}_q^{\otimes n}$ is the algebra generated by $\{A_r, B_r, C_r, D_r\}_{r \geq 0}$ subject to the commutation relations imposed by the Yang-Baxter equation (3.18).

Let u, v be two independent variables then one easily checks that (3.18) entails the relations

$$\mathcal{O}(u)\mathcal{O}(v) = \mathcal{O}(v)\mathcal{O}(u), \quad \mathcal{O} = A, B, C, D \quad (3.28)$$

and setting $(t = q^2)$

$$(u - v)A(u)B(v) = (ut - v)B(v)A(u) + (1 - t)vB(u)A(v), \quad (3.29)$$

$$(u - v)D(u)B(v) = (u - tv)B(v)D(u) - (1 - t)vB(u)D(v), \quad (3.30)$$

$$C(u)B(v) - tB(v)C(u) = \frac{(1 - t)v}{u - v} [A(v)D(u) - A(u)D(v)]. \quad (3.31)$$

Proposition 3.9 Introduce the co-product $\Delta : \mathfrak{A} \rightarrow \mathfrak{A} \times \mathfrak{A}$,

$$\begin{aligned} \Delta A(x) &= A(x) \otimes A(x) + C(x) \otimes B(x), \\ \Delta B(x) &= B(x) \otimes A(x) + D(x) \otimes B(x), \\ \Delta C(x) &= A(x) \otimes C(x) + C(x) \otimes D(x), \\ \Delta D(x) &= B(x) \otimes C(x) + D(x) \otimes D(x) \end{aligned} \quad (3.32)$$

and the co-unit $\varepsilon : \mathfrak{A} \rightarrow \mathbb{C}$,

$$\varepsilon(A) = \varepsilon(D) = 1 \quad \text{and} \quad \varepsilon(C) = \varepsilon(B) = 0.$$

Then $(\mathfrak{A}, \Delta, \varepsilon)$ is a bialgebra. That is, we have the identities

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta \quad \text{and} \quad (\varepsilon \otimes 1)\Delta = (1 \otimes \varepsilon)\Delta \cong 1.$$

Proof. All stated bi-algebra axioms are easily checked via a straightforward computation. ■

Lemma 3.10 We have the dependencies

$$(1 - t)B(u) = u A(u)\beta_1^* - ut\beta_1^*A(u) = D(u)\beta_n^* - t\beta_n^*D(u) \quad (3.33)$$

$$(1 - t)uC(u) = u\beta_n A(u) - utA(u)\beta_n = \beta_1 D(u) - tD(u)\beta_1 \quad (3.34)$$

Proof. The assertion is easily proved via induction employing the coproduct (3.32). ■

In analogy with (3.26) decompose the monodromy matrix as the sum of the form $T'(u) = \sum_{X,Y} X(u) \otimes Y$ with $X(u) \in \text{End } \mathcal{F}((u))$, $Y \in \mathcal{H}_q^{\otimes n}$ and consider the matrix elements of the monodromy matrix, $T'_{m'm}(u) := \sum_{X,Y} \langle m' | X(u) | m \rangle Y$. Then as a direct consequence of (3.24) we have the following identities.

Corollary 3.11 One deduces the following commutation relations of $T'_{m'm}(u)$ with the Yang-Baxter algebra generators,

$$\begin{aligned} (ut^{m'} + v)A(u)T'_{m'm}(v) - (ut^m + v)T'_{m'm}(v)A(u) = \\ v(1 - t^m)T'_{m',m-1}(v)B(u) - uC(u)T'_{m'-1,m}(v) \end{aligned} \quad (3.35)$$

$$T'_{m'-1}(v)B(u) = (ut^{m'} + v)B(u)T'_{m'-1}(v) - uT'_{m'm}(v)A(u) + uD(u)T'_{m'-1,m-1}(v) \quad (3.36)$$

$$\begin{aligned} uC(u)T'_{m'-1,m}(v) &= (ut^m + v)T'_{m'-1,m}(v)C(u) \\ &\quad - v(1 - t^m)T'_{m'-1,m-1}(v)D(u) + v(1 - t^{m'})A(u)T'_{m'm}(v) \end{aligned} \quad (3.37)$$

$$uD(u)T'_{m'm}(v) - uT'_{m'm}(v)D(u) = uT'_{m'+1,m}(v)C(u) - v(1 - t^{m'})B(u)T'_{m'+1,m}(v) \quad (3.38)$$

3.4 Affine quantum plactic polynomials

We now state simple polynomial expressions for the matrix elements of both monodromy matrices, (3.26) and (3.27), in the generators of an algebra $\hat{\mathcal{U}}_n^- \subset h_z(\hat{U}_q \mathfrak{b}^-)$ which is contained in the image of the lower Borel algebra $\hat{U}_q \mathfrak{b}^- \subset \hat{U}_n$ under the homomorphism (3.8).

Corollary 3.12 *Denote by $\hat{\mathcal{U}}_n^- \subset \mathcal{H}_q^{\otimes n}$ the subalgebra generated by the letters*

$$a_i = \beta_{i+1}^* \beta_i, \quad i = 1, \dots, n-1 \quad \text{and} \quad a_n = z \beta_1^* \beta_n. \quad (3.39)$$

Then we have non-local commutativity,

$$a_i a_j = a_j a_i \quad \text{for} \quad |i - j| \bmod n > 1, \quad (3.40)$$

and what we call the “quantum Knuth relations” ($t = q^2$)

$$\begin{aligned} a_{i+1} a_i^2 + t a_i^2 a_{i+1} &= (1 + t) a_i a_{i+1} a_i, \\ a_{i+1}^2 a_i + t a_i a_{i+1}^2 &= (1 + t) a_{i+1} a_i a_{i+1}, \end{aligned} \quad (3.41)$$

where all indices are understood modulo n . Denote by \mathcal{U}_n^- the non-affine algebra generated by $\{a_1, \dots, a_{n-1}\}$.

Proof. The assertion is a direct consequence of the quantum Serre relations (2.7) and the homomorphisms (3.7) and (3.8). ■

Remark 3.3 *Setting formally $t = 0$ we recover the faithful representation of the local affine plactic algebra considered in [40, Def 5.4 and Prop 5.8]. Setting $t = 0$ and $a_n = 0$ one obtains a representation of the local finite plactic algebra; compare with [19]. The (non-local) plactic algebra was introduced in [49] and is intimately linked to the Robinson-Schensted-Knuth correspondence [22]. We therefore shall refer to $\hat{\mathcal{U}}_n^-$ as affine quantum plactic algebra. Note that this definition is different from the construction in [46, Section 4.7] using the quantum coordinate ring.*

To describe the matrix elements of the monodromy matrices we require the notion of cyclically ordered words $i_1 \dots i_r$ with letters $i_j \in \{1, \dots, n\}$. A word $i_1 \dots i_r$ is *anti-clockwise cyclically ordered* if for any two indices i_j, i_k with $i_k = i_j + 1$ modulo n , the i_j occurs to the right of i_k . (In case $i_k \neq i_j + 1$ the order does not matter because of (3.40).) The origin of the name becomes obvious if we identify the letter i_j with labels of the nodes of the $\hat{\mathfrak{sl}}(n)$ Dynkin diagram: there are two circle segments connecting the two points. If they are

not of the same length we choose the shorter one and the anti-clockwise order is the same as the intuitively defined anti-clockwise order with respect to this segment. For any word $i_1 \dots i_r$ not containing all the letters in $\{1, \dots, n\}$, there is a unique anti-clockwise cyclically ordered word which differs only by a permutation; compare with [40, Section 5.3].

Proposition 3.13 (noncommutative elementary symmetric polynomials) *One has the identity*

$$E(u) := A(u) + zD(u) = \sum_{r=0}^n u^r e_r, \quad (3.42)$$

where we set $e_n = z$ and for $r < n$

$$e_r = (1-t)^{1-r} \sum_{w=i_1 \dots i_r} [a_{i_1}, [a_{i_2}, \dots [a_{i_{r-1}}, a_{i_r}]_{t \dots}]_t]_t \quad (3.43)$$

with $[x, y]_t := xy - t yx$ and the sum runs over all cyclically ordered words w with distinct elements.

Example 3.1 Set $n = 4$ and $r = 3$, then the affine quantum plactic elementary symmetric polynomial reads

$$(1-t)^2 e_r = [a_3, [a_2, a_1]_t]_t + [a_4, [a_3, a_2]_t]_t + [a_1, [a_4, a_3]_t]_t + [a_2, [a_1, a_4]_t]_t.$$

Remark 3.4 In the commutative case, making the formal replacement $a_i \rightarrow x_i$ with $x_i x_j = x_j x_i$ for all i, j , the noncommutative polynomial e_r becomes the standard, commutative elementary symmetric polynomial.

Proof. Employing the coproduct of the Yang-Baxter algebra one easily verifies via induction the formulae

$$\begin{aligned} A(u) &= \sum_{0 \leq r \leq \frac{n}{2}} \sum_{1 \leq i_1 < j_1 < \dots < i_r < j_r \leq n} u^{(j_1 - i_1) + \dots + (j_r - i_r)} a_{i_1, j_1} \dots a_{i_r, j_r}, \\ D(u) &= \sum_{0 \leq r \leq \frac{n}{2}} \sum_{1 \leq i_1 < j_1 < \dots < i_r < j_r \leq n} u^{n - (j_1 - i_1) - \dots - (j_r - i_r)} a_{j_1, i_1} \dots a_{j_r, i_r}, \end{aligned}$$

where $a_{i,j} := \beta_i \beta_j^*$. The remainder of the proof is now a consequence of the following lemma which can be proved via a straightforward computation.

Lemma 3.14 *For $i < j$ we have*

$$a_{i,j} = (1-q)^{i-j+1} [a_{j-1}, [a_{j-2}, \dots [a_{i+1}, a_i]_{t \dots}]_t]_t \quad (3.44)$$

and

$$a_{j,i} = z^{-1} (1-q)^{i-j-1} [a_{i-1}, [a_{i-2}, \dots [a_1, [a_n, [a_{n-1}, \dots [a_{j+1}, a_j]_{t \dots}]_t]_t]_t]_t. \quad (3.45)$$

■

Also the matrix elements of the other monodromy matrix (3.27) can be expressed as noncommutative analogues of a family of symmetric functions; compare with (2.17) and (2.18).

Proposition 3.15 (noncommutative Rogers-Szegö polynomials) *One has the formal power series expansion*

$$\mathbf{G}'(u) := \sum_{m \geq 0} z^m T'_{m,m} = \sum_{r \geq 0} u^r \mathbf{g}'_r, \quad \mathbf{g}'_r := \sum_{\lambda \vdash r} \frac{\mathbf{m}_\lambda}{(t)_\lambda}, \quad (3.46)$$

where the quantum plactic analogues of the symmetric monomial functions² are defined for partitions λ of length $\leq n$ as

$$\mathbf{m}_\lambda := \sum_w (z\beta_1^*)^{\lambda_{w_n}} a_1^{\lambda_{w_1}} \cdots a_{n-1}^{\lambda_{w_{n-1}}} \beta_n^{\lambda_{w_n}} \quad (3.47)$$

with the last sum running over all distinct permutations of λ .

Proof. According to its definition it is easy to verify that the matrix elements $T'_{m',m}(u) \in \mathbb{C}[[u]] \otimes \mathcal{H}_q^{\otimes n}$ of the monodromy matrix can be written as

$$T'_{m',m}(u) = \frac{u^m}{(t)_m} \sum_{m_1, \dots, m_{n-1} \geq 0} \frac{u^{m_1 + \dots + m_{n-1}}}{(t)_{m_1} \cdots (t)_{m_{n-1}}} (\beta_1^*)^m a_1^{m_1} a_2^{m_2} \cdots a_{n-1}^{m_{n-1}} \beta_n^{m'}. \quad (3.48)$$

Setting $m = m'$ the assertion follows. ■

Note that despite the sum in (3.46) being infinite, only a finite number of terms survive when acting on a vector in $\mathcal{F}^{\otimes n}$. Thus, the “generating function” \mathbf{G}' , which is a formal series here, becomes a well defined operator in $\text{End}(\mathcal{F}^{\otimes n})$; see (6.9) below.

Corollary 3.16 (Integrability) *All of the quantum plactic symmetric polynomials defined above commute pairwise, i.e. for any $r, r' \geq 0$ we have*

$$e_r e_{r'} = e_r e_{r'}, \quad \mathbf{g}'_r \mathbf{g}'_{r'} = \mathbf{g}'_{r'} \mathbf{g}'_r, \quad e_r \mathbf{g}'_{r'} = \mathbf{g}'_{r'} e_r. \quad (3.49)$$

This statement remains in particular true for $z = 0$, that is, for the finite plactic polynomials obtained by setting formally $a_n \equiv 0$ in e_r and considering only partitions λ with length $< n$ in (3.47) to compute \mathbf{g}'_r .

Proof. This is a direct consequence of the Yang-Baxter equations (3.18), (3.22), (3.24) and observing that the R -matrices (3.19), (3.21), (3.23) are compatible with quasi-periodic boundary conditions,

$$[R, \sigma \otimes 1 + 1 \otimes \sigma] = [R', N \otimes 1 + 1 \otimes N] = [R'', \sigma \otimes 1 + 1 \otimes N] = 0,$$

where $\sigma = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $N|m\rangle = m|m\rangle$ is the occupation or number operator. Using the above relations one shows with the help of (3.18) that

$$z^{\sigma_1} T_1(u) z^{\sigma_2} T_2(v) = R_{12}^{-1}(u/v) z^{\sigma_2} T_2(v) z^{\sigma_1} T_1(u) R_{12}(u/v)$$

and, hence, $\mathbf{E}(u)\mathbf{E}(v) = \mathbf{E}(v)\mathbf{E}(u)$ after taking the partial trace of the monodromy matrices on both sides. Similarly, one derives with the help of (3.22) and (3.24) that $\mathbf{G}'(u)\mathbf{G}'(v) = \mathbf{G}'(v)\mathbf{G}'(u)$ and $\mathbf{E}(u)\mathbf{G}'(v) = \mathbf{G}'(v)\mathbf{E}(u)$. Making a power series expansion with respect to the variables u, v and comparing coefficients on both sides, the assertion follows.

²Note that these polynomials do not commute in general.

The proof of the statement for $z = 0$ is now an immediate consequence of the definitions (3.42) and (3.46). Alternatively, it follows from the first relation (3.35) in Corollary 3.11 which yields $A(u)T'_{0,0}(v) = T'_{0,0}(v)A(u)$ when setting $m = m' = 0$. Moreover, one easily deduces from (3.18) and (3.22) that $A(u)A(v) = A(v)A(u)$ and $T'_{0,0}(u)T'_{0,0}(v) = T'_{0,0}(v)T'_{0,0}(u)$. ■

Returning briefly to the case of commuting variables, we can expect a functional relationship between the generating functions

$$G'(u) = \prod_{i>0} \frac{1}{(ux_i; t)_\infty} \quad \text{and} \quad E(u) = \prod_{i>0} (1 + ux_i) .$$

Namely, one verifies without difficulty that $G'(u)E(-u) = G'(ut)$ has to hold by formally manipulating the infinite products in G' . The following proposition states the noncommutative analogue of this relation which due to the periodic boundary conditions, i.e. the use of the affine instead of the finite quantum plactic algebra, contains an additional term.

Proposition 3.17 (functional equation) *The following functional equation in $\mathcal{H}_q^{\otimes n} \otimes \mathbb{C}[[u]]$ is valid,*

$$E(-u)G'(u) = G'(uq^2) + z(-u)^n G'(uq^{-2}) \prod_{i=1}^n q^{2N_i} . \quad (3.50)$$

Proof. We start by considering the kernel $W \subset \mathbb{C}^2 \otimes \mathcal{F}$ of $\tilde{R}(-1)$. The latter is spanned by the vectors $w_0 = |0, 0\rangle$ and $w_m = |0, m\rangle + |1, m-1\rangle$ for $m > 0$. Here $|\sigma, m\rangle = |\sigma\rangle \otimes |m\rangle$ and we have adopted the bra-ket notation also for \mathbb{C}^2 with $\sigma = 0, 1$. From the Yang-Baxter equation (3.24) we infer that

$$L_{13}(-u)L'_{23}(u)W \otimes \mathcal{H}_q \otimes \mathbb{C}[[u]] \subset W \otimes \mathcal{H}_q \otimes \mathbb{C}[[u]] .$$

A straightforward computation yields that for any $X \in \mathcal{H}_q \otimes \mathbb{C}[[u]]$

$$\begin{aligned} L_{13}(-u)L'_{23}(u)w_m \otimes X &= \\ u^m \sum_{m' \geq 0} \left[q^{2m} |0, m'\rangle \otimes \frac{\beta^{*m} \beta^{m'}}{(q^2)_m} X + |1, m'\rangle \otimes \left(\frac{\beta^{*m} \beta^{m'}}{(q^2)_m} - \frac{\beta^{*(m-1)} \beta^{m'}}{(q^2)_{m-1}} \right) X \right] &= \\ &= u^m q^{2m} \sum_{m' \geq 0} w_{m'} \otimes \frac{\beta^{*m} \beta^{m'}}{(q^2)_m} X, \end{aligned}$$

where we have used in the second line that $\beta\beta^{*m} = (1 - q^{2m})\beta^{*(m-1)} + \beta^{*m}\beta$; see (3.3). Thus, we can identify the action $L_{13}(-u)L'_{23}(u)$ on $W \otimes \mathcal{H}_q \otimes \mathbb{C}[[u]]$ with the action of $L'(uq^2)$ on $\mathcal{F}[[u]] \otimes \mathcal{H}_q$ using the isomorphism $W \cong \mathcal{F}$ with $w_m \mapsto |m\rangle$.

Let us now turn to the quotient space $(\mathbb{C}^2 \otimes \mathcal{F})/W$ which is spanned by the vectors $\bar{w}_m := |1, m\rangle$ with $m \geq 0$. We again calculate the action on the basis vectors for any $X \in \mathcal{H}_q \otimes \mathbb{C}[[u]]$ and find

$$\begin{aligned} L_{13}(-u)L'_{23}(u)\bar{w}_m \otimes X = & -\frac{u^{m+1}(1-q^{2m+2})}{(q^2)_m} \sum_{m' \geq 0} w_{m'} \otimes \beta^{*(m+1)}\beta^{m'} X - \frac{u^{m+1}}{(q^2)_m} \sum_{m' \geq 0} \bar{w}_{m'} \otimes \beta^{*m}\beta^{m'} X \\ & + \frac{u^{m+1}(1-q^{2m+2})}{(q^2)_m} \sum_{m' > 0} \bar{w}_{m'-1} \otimes \beta^{*(m+1)}\beta^{m'} X = \\ & -\frac{u^{m+1}}{(q^2)_m} \sum_{m' \geq 0} q^{-2m'} \bar{w}_{m'} \otimes \beta^{*m}\beta^{m'} q^{2N} X + \dots \end{aligned}$$

where in the last line we have omitted terms in $W \otimes \mathcal{H}_q$ since they are sent to zero under the quotient map. To arrive at this result use the relation $\beta^*\beta^{m'} = \beta^{m'-1}(1 - q^{2(N-m'+1)})$ which follows from (3.3). Using both formulae for the action of $L_{13}(-u)L'_{23}(u)$, one now easily shows that the trace of the product

$$T_0(-u)T'_{0'}(u) = L_{0n}(-u)L'_{0'n}(u) \cdots L_{01}(-u)L'_{0'1}(u),$$

with $0, 0'$ now labelling the factors in $\mathbb{C}^2 \otimes \mathcal{F}$ and $i = 1, \dots, n$ the factors in $\mathcal{H}_q^{\otimes n}$, can be written as the sum of the traces of the monodromy matrices $T'(uq^2)$ and $(-u)^n T'(uq^{-2})q^{2(N_1+\dots+N_n)}$. Hence, the assertion follows. ■

3.5 Noncommutative Macdonald polynomials and Cauchy identities

Employing known formulae for commutative symmetric functions we now define polynomials in the generators of the quantum plactic algebra $\hat{\mathcal{U}}_n^-$ which can be interpreted as noncommutative analogues of the Macdonald functions P'_λ, Q'_λ ; see (2.26).

Let $K(t) = M(s, P)$ be the unitriangular transition matrix between Schur and Hall-Littlewood P -functions and denote by $M(P, s) = K(t)^{-1}$ its inverse.

Definition 3.3 *We define the following noncommutative analogues of Macdonald polynomials*

$$P'_{\lambda'} := \sum_{\mu \geq \lambda} s_{\mu'} K_{\mu\lambda}(t), \quad s_\lambda := \det(e_{\lambda'_i - i + j})_{1 \leq i, j \leq \ell} \quad (3.51)$$

and

$$Q'_\lambda = \sum_{\mu' \leq \lambda'} K_{\lambda'\mu'}^{-1}(t) S'_{\mu'}, \quad S'_\lambda := \det(g'_{\lambda'_i - i + j})_{1 \leq i, j \leq \ell}, \quad (3.52)$$

where s_λ is the noncommutative Schur polynomial and S'_λ its noncommutative dual; see (2.30).

The matrix elements $K_{\mu\lambda}(t)$ are the celebrated Kostka-Foulkes polynomials for which explicit formulae are known; see e.g. [53, III.6 p242 and Examples 4, 7, pp 243-245] and references therein. The matrix elements of the inverse matrix, $K^{-1}(t)$, and hence Q'_λ can also be explicitly computed: denote by R_{ij} the familiar

raising and lowering operators of the ring of symmetric functions, $R_{ij}\lambda = (\lambda_1, \dots, \lambda_i + 1, \dots, \lambda_j - 1, \dots)$. Define $R'_{ji}F_\lambda := F_{(R_{ji}\lambda)^\vee}$, then

$$Q'_\lambda := \prod_{\lambda'_i > \lambda'_j} (1 - tR'_{ji})S'_\lambda$$

compare with [53, III.6].

Remark 3.5 *Note that the polynomials s_λ, S'_λ and Q'_λ are well-defined because of (3.49). In particular, we have for any λ, μ that*

$$S'_\lambda S'_\mu = S'_\mu S'_\lambda, \quad s_\lambda s_\mu = s_\mu s_\lambda, \quad S'_\lambda s_\mu = s_\mu S'_\lambda. \quad (3.53)$$

Note that in general some of the relations are different from the case of the usual symmetric functions in commuting variables: the functional relation (3.50) implies that

$$\sum_{a+b=c} (-1)^a e_a g'_b = t^c g'_c + zt^{N_{\text{tot}}} g'_{c-n} \quad (3.54)$$

with $N_{\text{tot}} = N_1 + \dots + N_n$. Only if we set $z = 0$ the above relation coincides with the known one for commuting variables, which is easily derived with help of the generating functions (2.15) and (2.18). Thus, in general we have that $P'_\lambda \neq b_{\lambda'}(t)Q'_\lambda$ which is different from the commutative case.

In contrast, the case $z = 0$ corresponding to the finite quantum plactic algebra has relations completely analogous to the commutative case, that is, the finite quantum plactic polynomials s_λ, S'_λ and P'_λ, Q'_λ with a_n set formally to zero behave just as their commutative counterparts. In particular, it then follows from the above definitions that for λ being a horizontal or vertical r -strip one has $Q'_{(r)} = g'_r$ and $(1-t)Q'_{(1^r)} = e_r$.

The definition of the above noncommutative polynomials is motivated by the following noncommutative analogues of Cauchy identities; compare with (2.32) and (2.33).

Corollary 3.18 (noncommutative Cauchy identities) *We have the expansions*

$$E(x_1) \cdots E(x_k) = \sum_{\lambda} m_{\lambda}(x) e_{\lambda} = \sum_{\lambda} s_{\lambda}(x) s_{\lambda'} = \sum_{\lambda} P_{\lambda}(x; t) P'_{\lambda'} \quad (3.55)$$

where $e_{\lambda} := e_{\lambda_1} e_{\lambda_2} \cdots$ and

$$G'(x_1) \cdots G'(x_{n-1}) = \sum_{\lambda} m_{\lambda}(x) g'_{\lambda} = \sum_{\lambda} s_{\lambda}(x) S'_{\lambda} = \sum_{\lambda} P'_{\lambda}(x; t) Q'_{\lambda}, \quad (3.56)$$

where $g'_{\lambda} := g'_{\lambda_1} g'_{\lambda_2} \cdots$.

Proof. The first equalities in (3.55) and (3.56) are obvious and follow from the definition of e_r and g'_r as the expansion coefficients of the matrices (3.42) and (3.46), respectively. To prove the other identities note that the definitions of $s_{\lambda'}, P'_{\lambda'}$ and $S'_{\lambda}, Q'_{\lambda}$ are the same as in the case of commuting variables. Thus, the expansions follow from the same arguments as in the case of commuting variables using the known transition matrices $M(s, m) = K^{-1}(1)$ [53, III.6, Table on page 241] and $M(P, s) = K(t)^{-1}$. ■

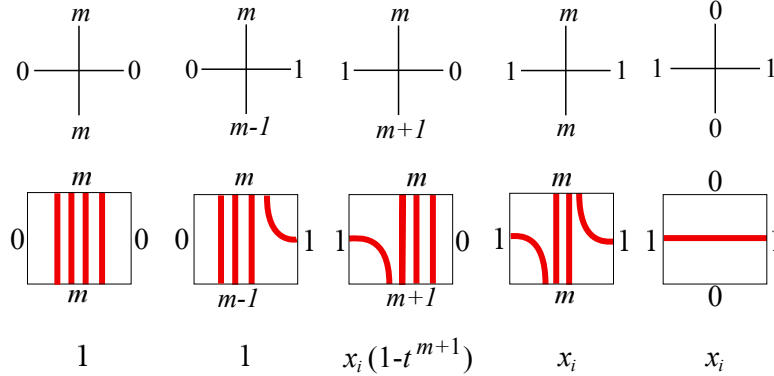


Figure 4.1: Depiction of the allowed vertex configurations associated with the L -operator (3.17); see top row. Horizontal edges can have values 0 and 1 only, while vertical edges can have values $m \in \mathbb{Z}_{\geq 0}$. Note the constraint that the sum of the values at the N and E edge equals the sum of the W and S values. Below, in the middle row, the vertex configurations are described in terms of nonintersecting paths. Listed at the bottom are the associated weights where i is the row index of the lattice.

4 Hall-Littlewood functions and the Yang-Baxter algebra

We now describe the action of the Yang-Baxter algebra employing the language of statistical mechanics: we show that the generators A, B, C, D can be used to compute partition functions of certain statistical vertex models defined on a finite square lattice. These partition functions turn out to be skew Hall-Littlewood functions. More precisely, the Yang-Baxter algebra generators play the role of row-to-row transfer matrices, i.e. their matrix elements yield the partition functions of a single lattice row with appropriate boundary conditions imposed on the lattice. We state for each generator a bijection between lattice configurations of the respective statistical model and (semi-standard) skew tableaux, showing that the Boltzmann weights of the statistical model coincide with the coefficient functions (2.20) and (2.21) appearing in the definition of skew Hall-Littlewood functions. This section can be seen as a preparatory step to motivate our construction of cylindric Hall-Littlewood functions in the subsequent section.

4.1 The statistical vertex model associated with L

Set $\mathbb{I}_r = \{0, 1, 2, \dots, r+1\} \subset \mathbb{Z}$ and consider the set $\mathbb{L} := \mathbb{I}_\ell \times \mathbb{I}_n$. We call $\dot{\mathbb{L}} := \{(i, j) \in \mathbb{L} \mid i \neq 0, \ell+1 \text{ and } j \neq 0, n+1\}$ the set of *interior lattice points* and $\partial\mathbb{L} := \mathbb{L} \setminus \dot{\mathbb{L}}$ the set of *boundary lattice points*, identifying $\dot{\mathbb{L}}$ with a square lattice of ℓ lattice rows and n lattice columns; see the definition below. The i th lattice row with $i = 1, \dots, \ell$ is the set $\{(r, s) \in \mathbb{L} : r = i\}$ and the j th lattice column with $j = 1, \dots, n$ is the set $\{(r, s) \in \mathbb{L} : s = j\}$. We will keep the number of lattice columns fixed throughout our discussion, while ℓ can vary.

Definition 4.1 (lattice edges) Let $u, v \in \mathbb{L}$. Then we call the pair

- (u, v) a horizontal edge if $u_1 = v_1$ and $u_2 + 1 = v_2$. We refer to u and v as the start and end point of

the edge, respectively. Denote by $\mathbb{E}_h \subset \mathbb{L} \times \mathbb{L}$ the set of all horizontal edges which start or end at an interior lattice point.

- (u, v) a vertical edge if $u_1 + 1 = v_1$ and $u_2 = v_2$. Similar as before we call u and v the start and end point and denote by $\mathbb{E}_v \subset \mathbb{L} \times \mathbb{L}$ the set of all vertical edges which start or end at an interior lattice point.

We call the horizontal and vertical edges which start (end) at a boundary lattice point and end (start) at an interior point *outer horizontal and vertical edges*, respectively.

We will often refer to the horizontal edges in the i th lattice row ($i = 1, \dots, \ell$) or the vertical edges in the j th lattice column ($j = 1, \dots, n$). By this we shall mean horizontal edges which start or end at a point $\langle i, r \rangle$ and vertical edges which start or end at a point $\langle s, j \rangle$, respectively. By the horizontal edges in the j th lattice column we shall mean those edges which end in the j th lattice column. Similarly the upper and lower vertical edges in the i th lattice row will be those vertical edges which respectively end and start in the i th row.

We now assign so-called statistical variables to the lattice edges.

Definition 4.2 (lattice & vertex configurations) A horizontal and vertical edge configuration are maps $\gamma_h : \mathbb{E}_h \rightarrow \{0, 1\}$ and $\gamma_v : \mathbb{E}_v \rightarrow \mathbb{Z}_{\geq 0}$, respectively. A pair $\gamma = (\gamma_h, \gamma_v)$ is simply called a lattice configuration. Given γ and a lattice point $\langle i, j \rangle \in \mathbb{L}$, we call the images of the horizontal and vertical edges which either start or end at this lattice point the vertex configuration at $\langle i, j \rangle$ and denote it by $\gamma_{\langle i, j \rangle}$.

Each lattice configuration γ can be assigned a statistical weight as follows: given a vertex configuration $\gamma_{\langle i, j \rangle} = \{\sigma, m, \sigma', m'\}$, where $\sigma, \sigma' = 0, 1$ are the images of the W and E horizontal edges and $m, m' \in \mathbb{Z}_{\geq 0}$ the images of the N and S vertical edges, define

$$\text{wt}(\gamma) := \prod_{\langle i, j \rangle \in \mathbb{L}} \text{wt}(\gamma_{\langle i, j \rangle}), \quad \text{wt}(\gamma_{\langle i, j \rangle}) := \langle \sigma, m | L(x_i) | \sigma', m' \rangle. \quad (4.1)$$

Here $\langle \sigma, m | L(x_i) | \sigma', m' \rangle$ are the matrix elements of the solution (3.17) to the Yang-Baxter equation with $\langle \sigma, m | \sigma', m' \rangle := \delta_{\sigma, \sigma'} \langle m | m' \rangle$ and x_i , $i = 1, \dots, \ell$ are some abstract commutative variables which only depend on the row index. The nonzero weights for the allowed vertex configurations are listed in Figure 4.1. Note that only the weights of vertex configurations for interior points contribute to the weight of a lattice configuration.

Remark 4.1 Alternatively, each lattice configuration can be described in terms of non-intersecting paths, closely related to the infinitely-friendly walker model in [44, 45], by mapping vertex configurations onto those path configurations shown in Figure 4.1. However, we will not employ this path picture in the proofs but instead continue to focus on an algebraic description in terms of vertex models.

In what follows we will impose *boundary conditions* on the lattice, that is we will prescribe certain values for the outer horizontal and vertical edges. Namely, given $\mu \in \mathcal{A}_{k,n}^+$, $\lambda \in \mathcal{A}_{k',n}^+$ and $\sigma, \tau = 0, 1$ consider the set $\Gamma_{\lambda, \mu}^{\sigma, \tau}$ of lattice configurations γ where the outer vertical edge starting at $\langle 0, j \rangle$ has value $m_j(\mu)$ and the

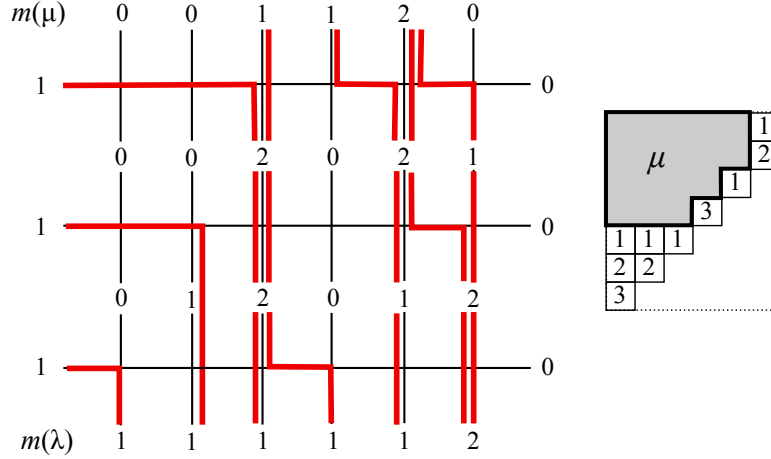


Figure 4.2: Graphical depiction of a sum over vertex configurations for the B -operator of the Yang-Baxter algebra (3.18) with $n = 6$, $k = 4$, $\ell = 3$ and $\mu = (5, 5, 4, 3)$, $\lambda = (6, 6, 5, 4, 3, 2, 1)$. Depicted on the right is the corresponding skew tableau under the bijection described in the proof of Lemma 4.2.

outer vertical edge ending at $\langle \ell + 1, j \rangle$ value $m_j(\lambda)$ for all $1 \leq j \leq n$. Fix the values of the outer horizontal edges either starting at $\langle i, 0 \rangle$ or ending at $\langle i, n + 1 \rangle$ to be σ or τ respectively for all $1 \leq i \leq \ell$.

Definition 4.3 (partition function) *The partition function is the weighted sum $Z_{\lambda, \mu}^{\sigma, \tau}(x_1, \dots, x_\ell) = \sum_{\gamma \in \Gamma_{\lambda, \mu}^{\sigma, \tau}} \text{wt}(\gamma)$ over those lattice configurations γ which satisfy the boundary conditions specified by $\lambda, \mu, \sigma, \tau$ as just described.*

Lemma 4.1 *We have the following identities between partition functions and matrix elements,*

$$Z_{\lambda, \mu}^{0,0}(x_1, \dots, x_\ell) = \langle \lambda | A(x_1) \cdots A(x_\ell) | \mu \rangle, \quad \lambda, \mu \in \mathcal{A}_{k,n}^+ \quad (4.2)$$

$$Z_{\lambda, \mu}^{1,0}(x_1, \dots, x_\ell) = \langle \lambda | B(x_1) \cdots B(x_\ell) | \mu \rangle, \quad \lambda \in \mathcal{A}_{k+\ell,n}^+, \mu \in \mathcal{A}_{k,n}^+ \quad (4.3)$$

$$Z_{\lambda, \mu}^{0,1}(x_1, \dots, x_\ell) = \langle \lambda | C(x_1) \cdots C(x_\ell) | \mu \rangle, \quad \lambda \in \mathcal{A}_{k-\ell,n}^+, \mu \in \mathcal{A}_{k,n}^+ \quad (4.4)$$

$$Z_{\lambda, \mu}^{1,1}(x_1, \dots, x_\ell) = \langle \lambda | D(x_1) \cdots D(x_\ell) | \mu \rangle, \quad \lambda, \mu \in \mathcal{A}_{k,n}^+ \quad (4.5)$$

Proof. This is a direct consequence of the definition (4.1) and the monodromy matrix (3.26). ■

Example 4.1 *Figure 4.2 shows an example for an allowed configuration for the B -operator when $n = 6$, $k = 4$, $\ell = 3$ and $\mu = (5, 5, 4, 3)$, $\lambda = (6, 6, 5, 4, 3, 2, 1)$. Because in each row one path is entering on the outer horizontal edge from the left and none is exiting on the right, the level $k = \sum_{i=1}^n m_i$ increases by one in each row and, thus, the partition function is only nonzero for $\lambda \in \mathcal{A}_{k+\ell,n}^+$.*

Lattice-tableau bijection 1: the A and B -operator

Lemma 4.2 *Let $\lambda \in \mathcal{A}_{k+\ell,n}^+$ and $\mu \in \mathcal{A}_{k,n}^+$. There exists a bijection $\gamma \mapsto T(\gamma)$ between lattice configurations $\gamma \in \Gamma_{\lambda, \mu}^{1,0}$ and skew tableaux T of shape λ/μ such that $\text{wt}(\gamma) = \varphi_{T(\gamma)} x^{T(\gamma)}$, where φ_T is defined in (2.20). The analogous statement holds for $\lambda, \mu \in \mathcal{A}_{k,n}^+$ and $\gamma \in \Gamma_{\lambda, \mu}^{0,0}$.*

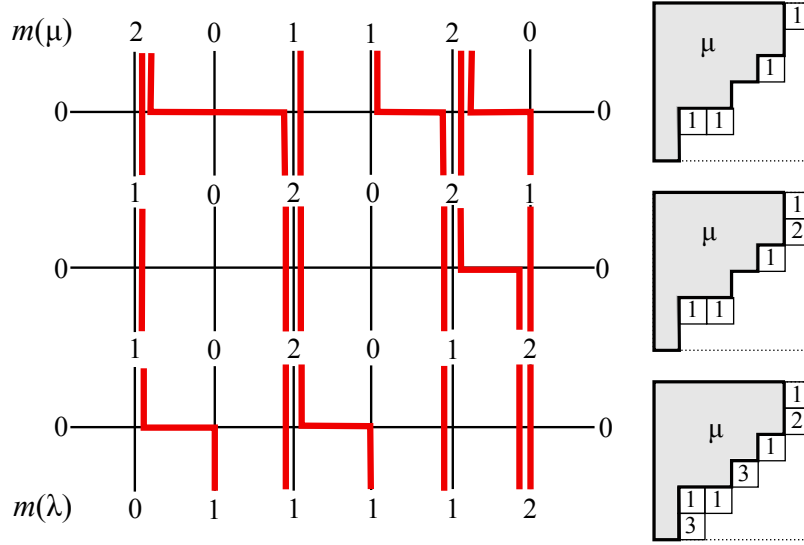


Figure 4.3: Graphical depiction of a sum over vertex configurations for the A -operator of the Yang-Baxter algebra (3.18) with $n = k = 6$, $\ell = 3$ and $\mu = (5, 5, 4, 3, 1, 1)$, $\lambda = (6, 6, 5, 4, 3, 2)$. Depicted on the right are the corresponding skew diagrams for each lattice row; see the proof of Lemma 4.2.

Proof. We will make use of the fact that a (semi-standard) tableau T of shape λ/μ is equivalent to a sequence of partitions $(\mu = \lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(\ell)} = \lambda)$ such that $\lambda^{(i+1)}/\lambda^{(i)}$ is a horizontal strip; see e.g. [53, Chapter I, Section 1]. It will therefore suffice to prove the bijection for a single horizontal strip and a single row configuration.

Assume $\ell = 1$, that is consider one lattice row only. Fix an allowed horizontal edge configuration $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_{n+1}) \in \{0, 1\}^{n+1}$ in $\Gamma_{\lambda, \mu}^{1,0}$. According to Figure 4.1 $m_i(\lambda) - m_i(\mu) = \sigma_i - \sigma_{i+1}$ for allowed configurations, where $m_i(\lambda) = \lambda'_i - \lambda'_{i+1}$ and $m_i(\mu) = \mu'_i - \mu'_{i+1}$. Moreover, we must have that $0 \leq \sigma_{i+1} \leq \min(1, m_i(\mu))$ with $\sigma_1 = 1$ and $\sigma_{n+1} = 0$. Hence,

$$\sigma_{i+1} = \sigma_i + m_i(\mu) - m_i(\lambda) = \sigma_1 + \sum_{j=1}^i (m_j(\mu) - m_j(\lambda)) \quad (4.6)$$

and the horizontal edge value σ_i in the i th lattice column is given by $\sigma_i = \lambda'_i - \mu'_i = 0, 1$ because $\mu'_1 = \sum_{j=1}^n m_j(\mu) = k$ while $\lambda'_1 = \sum_{j=1}^n m_j(\lambda) = k + 1$. Thus, λ/μ is a horizontal strip with a box in the i th column of the skew diagram if $\sigma_i = 1$.

Conversely, it is easy to see that each horizontal strip λ/μ with $\lambda_1, \mu_1 \leq n$ which has a box in the first column defines a unique allowed row configuration of the lattice employing the same formulae in reverse order.

Recall from the definition (2.20), that $\varphi_{\lambda/\mu}$ contains a factor $(1 - t^{m_i(\lambda)})$ if there is a box added in the i th column of the skew diagram but none in the $(i+1)$ th. In terms of the correspondence between row configurations and horizontal strips this means that the value of the horizontal edge in i th lattice column is one and zero in the $(i+1)$ th. Thus, we obtain the weight of the third vertex configuration shown in Figure 4.1. Finally, it is obvious that the sum $r = \sum_{i=1}^{n+1} \sigma_i$ over the horizontal edge values gives the length of the

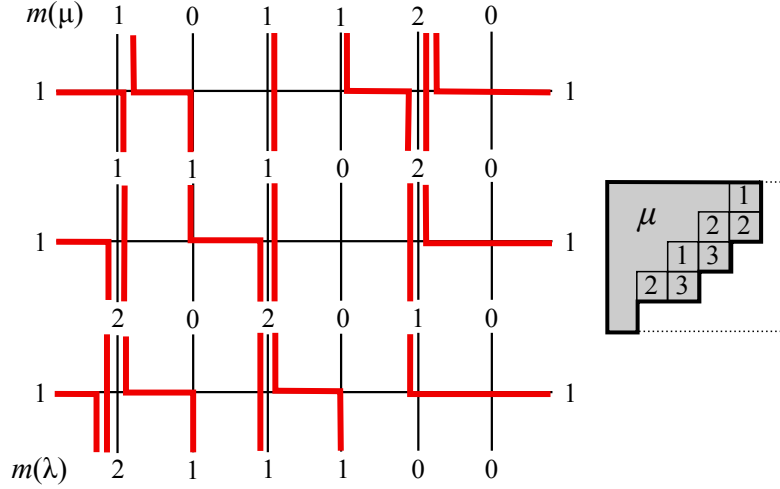


Figure 4.4: Graphical depiction of a sum over vertex configurations for the D -operator of the Yang-Baxter algebra (3.18) with $n = 6$, $k = 5$, $\ell = 3$ and $\mu = (5, 5, 4, 3, 1)$, $\lambda = (4, 3, 2, 1, 1)$. Depicted on the right is the corresponding skew tableau; see the proof of Lemma 4.3.

horizontal strip. From this we now easily deduce that $\text{wt}(\gamma(T)) = \varphi_{\lambda/\mu} x^r$ as desired.

Assume now $\ell \geq 1$ then it follows that a lattice configuration defines a sequence of horizontal strips of the type just described and hence we end up with skew tableaux of λ/μ with ℓ boxes in the first column: starting from the first lattice row on the top add a box labelled with one to each column of the Young diagram of μ whenever a horizontal path edge occurs in the lattice column with the same number. Then continue and do the same for the second lattice row labeling boxes now with 2, and so on. Conversely, given a skew tableau T of shape λ/μ with $\lambda \in \mathcal{A}_{k+\ell, n}^+$ and $\mu \in \mathcal{A}_{k, n}^+$, the first column of the skew tableau T of shape λ/μ must be of height ℓ with boxes labelled from 1 to ℓ . It thus decomposes into a sequence of allowed row configurations. The equality between the weight $\text{wt}(\gamma)$ of the lattice configuration $\gamma = \gamma(T)$ corresponding to the tableau T and the value of the coefficient function φ_T is now a direct consequence of the fact that $\varphi_T = \prod_{i \geq 0} \varphi_{\lambda^{(i+1)}/\lambda^{(i)}}$ with $\lambda^{(i+1)}/\lambda^{(i)}$ being the horizontal strips determined by T and that the weight $\text{wt}(\gamma)$ of the lattice configuration γ is the product of the weights of its row configurations, see (4.1).

The construction of the bijection for $\lambda, \mu \in \mathcal{A}_{k, n}^+$ and $\gamma \in \Gamma_{\lambda, \mu}^{0, 0}$ is completely analogous and only differs in the boundary conditions imposed on the square lattice: now the left *and* the right boundary edges are set to zero, so one has $\sigma_1 = \sigma_{n+1} = 0$ in each row configuration. An example is shown in Figure 4.3. From the graphical depiction of the weights of vertex configurations in Figure 4.1 it is evident that these boundary conditions imply that the level $k = \sum_{i=1}^n m_i(\lambda)$ is preserved in each lattice row: the sum over the values of the outer vertical edges on the top must equal the sum of the values of the outer vertical edges at the bottom. This means that we have to choose λ and $\mu \in \mathcal{A}_{k, n}^+$ to obtain allowed lattice configurations. Conversely, only skew diagrams λ/μ which have no boxes in the first column correspond to such lattice configurations. We omit the remainder of the proof as it now closely follows along the previous lines. ■

Lattice-tableau bijection 2: the C and D -operator

Lemma 4.3 *Let $\lambda \in \mathcal{A}_{k-\ell,n}^+$ and $\mu \in \mathcal{A}_{k,n}^+$ with $\ell \leq k$. There exists a bijection $\gamma \mapsto T(\gamma)$ between lattice configurations $\gamma \in \Gamma_{\lambda,\mu}^{0,1}$ and Young tableaux T of shape μ/λ such that $\text{wt}(\gamma) = \psi_{T(\gamma)} x^{T(\gamma)}$. A similar statement holds for $\gamma \in \Gamma_{\lambda,\mu}^{1,1}$ and $\lambda, \mu \in \mathcal{A}_{k,n}^+$.*

Proof. We will discuss the D -operator case only, the generalization to the C -operator will then be obvious. The outer horizontal edges on the left and right boundary now all have value one, $\sigma = \tau = 1$; compare with the definition of the D -operator via (3.26). By the same reasoning as before it suffices to consider the simplest case $\ell = 1$.

Employing the relations $m_i(\lambda) - m_i(\mu) = \sigma_i - \sigma_{i+1}$, $0 \leq \sigma_{i+1} \leq \min(1, m_i(\mu))$ and $\sigma_1 = \sigma_{n+1} = 1$ for allowed row configurations according to Figure 4.1, we now infer with the help of (4.6) that $1 - \sigma_i = \mu'_i - \lambda'_i = 0, 1$ since $\lambda, \mu \in \mathcal{A}_{k,n}^+$. Thus, we obtain a horizontal strip μ/λ which has a box in the i th column of the skew diagram if $\sigma_i = 0$. Conversely, given a horizontal strip of shape μ/λ with $\lambda, \mu \in \mathcal{A}_{k,n}^+$ we must have $\sigma_1 = \sigma_{n+1} = 1$ since there can be no boxes in the first or $(n+1)$ th column.

The equality between weights $\text{wt}(\gamma)$ of row configurations γ and the coefficient function $\psi_{\mu/\lambda}$ defined in (2.21) follows again from their definitions: $\psi_{\mu/\lambda}$ contains a factor $(1 - t^{m_i(\lambda)})$ if the skew diagram μ/λ has a box in the $(i+1)$ th column but not in the preceding one. Comparison with the vertex configurations in Figure 4.1 shows that this matches again with the third vertex configuration and its weight. Furthermore, we obviously have that $r = n + 1 - \sum_{i=1}^{n+1} \sigma_i$, where σ_i are the horizontal edge values, equals the length of the horizontal strip and, thus, $\text{wt}(\gamma) = x_1^{n-r} \psi_{\mu/\lambda}$.

The general case of $\ell \geq 1$ is obtained again by writing each tableau T as a sequence of horizontal strips: label the lowest box *within* each column of the Young diagram of μ with ℓ whenever there is no horizontal edge in the corresponding lattice column. (We employ the same numbering convention of columns in the Young diagram and the lattice as before.) Do the same for the second lattice row labeling boxes now with $\ell - 1$. Continue with this procedure up to the last row. An example is provided in Figure 4.4. The equality of lattice configuration weights with $(x_1 \cdots x_\ell)^n \psi_T x^{-T}$ is once more immediate from the definition of ψ_T and the fact that lattice configurations are products of row configurations.

Finally, matrix elements for the C -operator are obtained by setting $\sigma_1 = 0$ and $\sigma_{n+1} = 1$ in each row. Allowed lattice configurations can therefore only occur if $\lambda \in \mathcal{A}_{k-\ell,n}^+$ and $\mu \in \mathcal{A}_{k,n}^+$ and the corresponding skew tableaux μ/λ will have ℓ boxes in the first column. The remainder of the proof now follows along the same lines as before. ■

4.2 Skew Hall-Littlewood functions as partition functions

The following proposition and corollary summarise the findings of the two previous lemmata with $\varphi_{\lambda/\mu}$ and $\psi_{\mu/\lambda}$ defined in (2.20) and (2.21), respectively.

Proposition 4.4 (Pieri-type formulae) *Let $\mu \in \mathcal{A}_{k,n}^+$ and set $t = q^2$. Then the action of the Yang-Baxter*

operators can be expressed as

$$A_r|\mu\rangle = \sum_{\substack{\lambda-\mu=(r), \\ \lambda \in \mathcal{A}_{k,n}^+}} \varphi_{\lambda/\mu}(t)|\lambda\rangle, \quad B_r|\mu\rangle = \sum_{\substack{\lambda-\mu=(r), \\ \lambda \in \mathcal{A}_{k+1,n}^+}} \varphi_{\lambda/\mu}(t)|\lambda\rangle \quad (4.7)$$

and

$$C_r|\mu\rangle = \sum_{\substack{\mu-\lambda=(n-r), \\ \lambda \in \mathcal{A}_{k-1,n}^+}} \psi_{\mu/\lambda}(t)|\lambda\rangle, \quad D_r|\mu\rangle = \sum_{\substack{\mu-\lambda=(n-r), \\ \lambda \in \mathcal{A}_{k,n}^+}} \psi_{\mu/\lambda}(t)|\lambda\rangle, \quad (4.8)$$

where the notation $\lambda - \mu = (r)$ and $\mu - \lambda = (n - r)$ means that the skew diagrams λ/μ and μ/λ are horizontal strips of length r and $n - r$, respectively. In particular we have $A_r^* = D_{n-r}$ and $B_r^* = C_{n-r}$ with respect to the inner product on the Fock space $\mathcal{F}^{\otimes n}$.

The multiple action of the Yang-Baxter algebra generators A, B, C, D can be described in terms of skew Hall-Littlewood functions.

Corollary 4.5 *Let $\mu \in \mathcal{A}_{k,n}^+$ and x_1, \dots, x_ℓ be some generic variables, then we have*

$$A(x_1) \cdots A(x_\ell)|\mu\rangle = \sum_{\lambda \in \mathcal{A}_{k,n}^+} Q_{\lambda/\mu}(x_1, \dots, x_\ell; t)|\lambda\rangle, \quad (4.9)$$

$$B(x_1) \cdots B(x_\ell)|\mu\rangle = \sum_{\lambda \in \mathcal{A}_{k+\ell,n}^+} Q_{\lambda/\mu}(x_1, \dots, x_\ell; t)|\lambda\rangle, \quad (4.10)$$

$$C(x_1) \cdots C(x_\ell)|\mu\rangle = (x_1 \cdots x_\ell)^n \sum_{\lambda \in \mathcal{A}_{k-\ell,n}^+} P_{\mu/\lambda}(x_1^{-1}, \dots, x_\ell^{-1}; t)|\lambda\rangle, \quad (4.11)$$

$$D(x_1) \cdots D(x_\ell)|\mu\rangle = (x_1 \cdots x_\ell)^n \sum_{\lambda \in \mathcal{A}_{k,n}^+} P_{\mu/\lambda}(x_1^{-1}, \dots, x_\ell^{-1}; t)|\lambda\rangle. \quad (4.12)$$

Remark 4.2 *Note in particular that for $|\emptyset\rangle := |(0, 0, \dots)\rangle$ (the pseudo-vacuum) we have the following expansion*

$$B(x_1) \cdots B(x_k)|\emptyset\rangle = \sum_{\lambda \in \mathcal{A}_{k,n}^+} Q_\lambda(x_1, \dots, x_k; t)|\lambda\rangle. \quad (4.13)$$

For this special case the connection with Hall-Littlewood functions has been first discussed in [72, Section 3, Propositions 3 and 4]. However, the formula (4.13) differs from the one in [72, Section 3, Propositions 3 and 4]: in our result the sums are restricted to partitions of a fixed length $\ell(\lambda) = k \geq 0$. Without this restriction one obtains erroneous results.

5 Cylindric Hall-Littlewood functions

So far we have considered lattice configurations where the horizontal edge values on the left and on the right boundary are fixed to certain values in each row. Now we wish to consider (quasi) periodic boundary conditions, i.e. we move onto the cylinder by identifying the first and the last lattice column.

5.1 Periodic boundary conditions

Given $\lambda, \mu \in \mathcal{A}_{k,n}^+$ we now restrict $1 \leq \ell \leq k$ and consider the set $\Gamma_{\lambda,\mu}$ of lattice configurations γ where the outer vertical edge starting at $\langle 0, j \rangle$ has value $m_j(\mu)$ and the outer vertical edge ending at $\langle \ell + 1, j \rangle$ value $m_j(\lambda)$ for all $1 \leq j \leq n$ as before. Moreover, γ_h should satisfy $\gamma_h(\langle i, 0 \rangle, \langle i, 1 \rangle) = \gamma_h(\langle i, n \rangle, \langle i, n + 1 \rangle)$ for all $1 \leq i \leq \ell$. That is, the values of the outer horizontal edges either starting at $\langle i, 0 \rangle$ or ending at $\langle i, n + 1 \rangle$ need to be the same. Clearly this condition is equivalent to considering the cylindric lattice $\mathbb{L}^{\text{cyl}} := \mathbb{L}_\ell \times \mathbb{Z}_n$ with the obvious generalisations of the definitions of horizontal, vertical edges and lattice configurations.

Denote by $\Gamma_{\lambda/d/\mu} := \{\gamma \in \Gamma_\lambda^\mu : \sum_{i=1}^\ell \gamma_h(\langle i, 0 \rangle, \langle i, 1 \rangle) = d\}$ the subset of configurations where the sum over the values of the left (or right) outer horizontal edges is d and let

$$Z_{\lambda/d/\mu}(x_1, \dots, x_\ell) := \sum_{\gamma \in \Gamma_{\lambda/d/\mu}} \text{wt}(\gamma) \quad (5.1)$$

be the partition function for the lattice with periodic boundary conditions in the horizontal direction. We introduce a formal variable z keeping track of the winding number around the cylinder; the latter is the same variable as the one used in (3.39) and (3.42).

Lemma 5.1 *We have the identity*

$$\langle \lambda | \mathbf{E}(x_1) \cdots \mathbf{E}(x_\ell) | \mu \rangle = \sum_{d \geq 0} z^d Z_{\lambda/d/\mu}(x_1, \dots, x_\ell), \quad \ell \leq k, \quad (5.2)$$

and, hence, in light of definition (3.42) that

$$Z_{\lambda/d/\mu}(x_1, \dots, x_\ell) = \sum_{0 < i_1 < \cdots < i_d \leq \ell} \langle \lambda | A(x_1) \cdots D(x_{i_1}) \cdots D(x_{i_d}) \cdots A(x_\ell) | \mu \rangle. \quad (5.3)$$

In particular, the partition function $Z_{\lambda/d/\mu}(x_1, \dots, x_\ell)$ is symmetric in the variables (x_1, \dots, x_ℓ) .

Proof. The first identity follows from the definition (3.42). Due to (3.49), which implies $\mathbf{E}(x_i)\mathbf{E}(x_j) = \mathbf{E}(x_j)\mathbf{E}(x_i)$, we know that the sum in (5.2) must be symmetric in (x_1, \dots, x_ℓ) . But since z is an independent formal variable (the weights of the lattice configurations do not depend on z) we can conclude that each single summand must be symmetric as well. ■

Remark 5.1 *The partition function (5.2) for $z = 1$ is related to the so-called q -boson model discussed by Bogoliubov, Izergin and Kitanine in [7]. This model is quantum integrable and the associated discrete Hamiltonians are*

$$H_r^\pm := -\frac{e_r \pm e_{n-r}}{2}, \quad 1 \leq r \leq n/2. \quad (5.4)$$

In particular, the operator

$$H_1^+ = -\frac{1}{2} \sum_{i \in \mathbb{Z}_n} (\beta_i \beta_{i+1}^* + \beta_i^* \beta_{i+1}), \quad (5.5)$$

describes nearest neighbour hopping of certain highly-correlated quantum particles, so-called q -bosons, on a one-dimensional periodic lattice (i.e. a circle) with n sites. The interaction between the particles is encoded in the q -deformation. This model can also be interpreted as a quantization of the Ablowitz-Ladik hierarchy in integrable systems. The latter describes a discrete version of the nonlinear Schrödinger model.

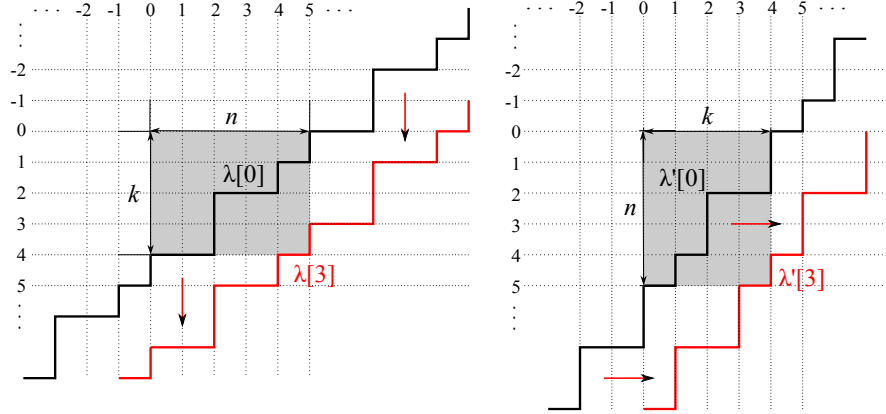


Figure 5.1: On the left are shown two cylindric loops constructed from the partition $\lambda = (5, 4, 2, 2)$. The loop $\lambda[0]$ is simply a periodic continuation of the outline of the Young diagram of λ . The loop $\lambda[3]$ is then obtained by shifting three times in the direction of the vector $(1, 0)$. On the right are the corresponding conjugate cylindric loops.

Corollary 5.2 *According to Proposition 4.4 the action of $\mathbf{E}(u) = \sum_{r \geq 0} u^r \mathbf{e}_r$ is given by*

$$\mathbf{e}_r |\mu\rangle = \sum_{\substack{\lambda - \mu = (r), \\ \lambda \in \mathcal{A}_{k,n}^+}} \varphi_{\lambda/\mu}(t) |\lambda\rangle + z \sum_{\substack{\mu - \lambda = (n-r), \\ \lambda \in \mathcal{A}_{k,n}^+}} \psi_{\mu/\lambda}(t) |\lambda\rangle. \quad (5.6)$$

and for $\bar{z} = z^{-1}$, $\bar{u} = u$ we have $\mathbf{E}^*(u) = z^{-1} u^n \mathbf{E}(u^{-1})$.

We now wish to derive an expansion of the partition function $Z_{\lambda/d/\mu}$ of the q -boson model on the cylinder, which is analogous to the formulae appearing in Corollary 4.5. Before we can do so, we need some additional combinatorial notions.

5.2 Cylindric loops and cylindric skew tableaux

Recall that a skew diagram can be seen as subset of \mathbb{Z}^2 , $\lambda/\mu := \{(i, j) \in \mathbb{Z}^2 : 1 \leq i \leq \ell(\lambda), \mu_i < j \leq \lambda_i\}$. Inspired by the discussion in [27], [56] and [54] we now define *cylindric skew diagrams*. The borders of such cylindric skew diagram are so-called cylindric loops, which are a periodic generalisation of the Ferrer's shapes or Young diagrams of ordinary partitions.

Definition 5.1 (cylindric loop) *Let $\lambda \in \mathcal{A}_{k,n}^+$ and $r \in \mathbb{Z}$. Define the associated cylindric loop $\lambda[r] = (\dots, \lambda[r]_{-1}, \lambda[r]_0, \lambda[r]_1, \lambda[r]_2, \dots)$ as the sequence obtained from $\lambda[r]_i := \lambda[0]_{i-r}$ with $\lambda[0]_i := \lambda_i$ for $1 \leq i \leq k$ and $\lambda[0]_{i+\ell} := \lambda[0]_i - \ell n$ outside this interval.*

In other words, $\lambda[0]$ is the path in $\mathbb{Z} \times \mathbb{Z}$ traced out when translating the outline of the Young diagram of λ by the period vector $\Omega = (k, -n)$. The loop $\lambda[r]$ is then its shift by the vector $(r, 0)$. The reason for the name “cylindric loop” should be apparent from the definition; see Figure 5.1 for an example.

Definition 5.2 (cylindric skew diagram) Let $\lambda, \mu \in \mathcal{A}_{k,n}^+$. Define the (n, k) -restricted cylindric skew diagram of degree $d \geq 0$ as the set

$$\lambda/d/\mu := \{ \langle i, j \rangle \in \mathbb{Z} \times \mathbb{Z} \mid \lambda[d]_i \geq j > \mu[0]_i \} . \quad (5.7)$$

A cylindric skew diagram which in each column (row) has at most one square is called a cylindric horizontal (vertical) strip.

Broadly speaking the cylindric skew diagram is the “periodic continuation” of the skew diagram $\Lambda^{(d)}/\mu$ by the period vector $\Omega = (k, -n)$, where $\Lambda^{(d)}$ is obtained by adding d parts of size n to λ . Letting $\lambda, \mu \in \mathcal{A}_{k,n}^+$ it follows that the cylindric diagram $\lambda/d/\mu$ can have at most k squares in each column and at most n elements in each row. Hence, we call such diagrams (n, k) -restricted³ and shall henceforth only consider such.

The *fundamental region* of a cylindric skew diagram $\lambda/d/\mu$ is the following finite set of squares between the two loops $\lambda[d]$ and $\mu[0]$,

$$[\lambda/d/\mu]_0 := \{ \langle i, j \rangle \in \mathbb{Z}^2 : 1 \leq i \leq k + d, \mu_i[0] < j \leq \lambda[d]_i, \ell \in \mathbb{Z} \} .$$

Conversely, given a (n, k) -restricted cylindric skew diagram Θ one can reconstruct the partitions $\lambda, \mu \in \mathcal{A}_{k,n}^+$ and degree d . It is easiest to explain this on a concrete example like the one shown in Figure 5.2.

One first identifies a fundamental region, that is one fixes a bounding box of height k and width n in the $\mathbb{Z} \times \mathbb{Z}$ lattice such that the squares contained in it generate the entire diagram upon shifting with $(k, -n)$. Then one defines μ to be the partition whose Young diagram is cut out by the bounding box and the upper boundary of the cylindric skew diagram. To obtain the partition λ consider first the partition Λ obtained by adding to μ all the boxes of the cylindric skew diagram within the specified n -strip. Then remove from Λ rows of size n until the associated Young diagram has height k , i.e. fits into the bounding box. The degree d is the number of n -rows removed. Note that for $d = 0$ we recover the familiar skew-diagram of two partitions, i.e. $[\lambda/0/\mu]_0 = \lambda/\mu$.

Remark 5.2 The set $\lambda/d/\mu$ is a particular way of parametrizing certain cylindric skew diagrams or cylindric skew shapes with period vector $\Omega = (k, -n)$; compare with the general definition of (reversed) cylindric plane partitions in [27] and [54, Section 3] for a discussion in terms of oriented posets. While we use the same notation as in [54, Section 4] and [56], our parametrization of a cylindric skew diagram in terms of two partitions λ, μ and a degree d differs from the one used in loc. cit. in order to accommodate the description of the Verlinde algebra given in [40].

For our discussion we will also need the transposed or *conjugate cylindric skew diagram* which we denote by $\lambda'/d/\mu'$ and is defined as the set

$$\lambda'/d/\mu' := \{ \langle i + \ell n, j - \ell k \rangle \in \mathbb{Z}^2 : 1 \leq i \leq n, \mu'_i < j \leq \lambda'_i + d, \ell \in \mathbb{Z} \} . \quad (5.8)$$

The latter can also be described in terms of *conjugate cylindric loops* $\lambda'[r]$. Consider first the case $r = 0$ and set $\lambda'[0]_i = \lambda'_i$ for $1 \leq i \leq n$ and $\lambda'[0]_{i+n} = \lambda_i - ik$ otherwise. The conjugate cylindric loop $\lambda'[r]$ is then

³Postnikov called such restricted cylindric shapes “toric”; see [56, Def 3.2] and [54, Paragraph after eqn (4.1) on page 289].

obtained by shifting $\lambda'[0]$ by the vector $(0, r)$, i.e. we set $\lambda'[r]_i = \lambda'[0]_i + r$. It is then straightforward to verify that $\lambda'/d/\mu' = \{\langle i, j \rangle \in \mathbb{Z}_n \times \mathbb{Z}_k \mid \lambda'[d]_i \geq j > \mu'[0]_i\}$ and that the transposed or conjugate cylindric skew diagram is obtained by transposing $\lambda/d/\mu$; see Figure 5.1. In the introduction we discussed these conjugate cylindric loops and skew diagrams.

Definition 5.3 (cylindric skew tableau) *Let Θ be a (n, k) -restricted cylindric skew diagram. A cylindric (semi-standard) skew tableau is a map $T : \Theta \rightarrow \mathbb{N}$ such that for any $\langle i, j \rangle \in \Theta$ one has*

$$T(i, j) = T(i + k, j - n), \quad (5.9)$$

$$T(i, j) < T(i + 1, j), \text{ if } \langle i + 1, j \rangle \in \Theta \quad (5.10)$$

$$T(i, j) \leq T(i, j + 1), \text{ if } \langle i, j + 1 \rangle \in \Theta. \quad (5.11)$$

The weight vector $\text{wt}(T) = (t_1, \dots, t_k)$ of a cylindric tableau is defined by setting t_i to be the number of i -entries in T in the fundamental region.

In other words, a tableau T of a (restricted) cylindric shape $\lambda/d/\mu$ is a filling of the squares of the associated diagram with integers such that in each row the numbers are weakly increasing (left to right), while they strictly increase (top to bottom) in each column; see Figure 5.2 for an example.

Definition 5.4 The cylindric Kostka number⁴ $K_{\lambda/d/\mu, \theta}$ is defined to be the number of cylindric tableaux of weight $\text{wt}(T) = \theta$, and specialises for $d = 0$ to the ordinary Kostka number $K_{\lambda/\mu, \theta}$.

Remark 5.3 Note that because of condition (5.11) not every ordinary skew tableau of shape $\Lambda^{(d)}/\mu$ (with $\Lambda^{(d)}$ being the partition obtained from λ by adding d parts of size n) gives rise to a cylindric tableau via periodic continuation with respect to the vector $\Omega = (k, -n)$. For instance, consider the ordinary skew tableau

					3
			1	1	4
			2	3	
	1	3	4		
	2				
	3				

which is of the shape of the skew diagram shown in the grey box in Figure 5.2. This tableau does not give rise to a cylindric skew tableau when periodically continued.

Lemma 5.3 Any cylindric skew tableau T of shape $\lambda/d/\mu$ with $\lambda, \mu \in \mathcal{A}_{k,n}^+$ is equivalent to a sequence of cylindric loops

$$(\lambda^{(0)}[d_0 = 0] = \mu[0], \lambda^{(1)}[d_1], \dots, \lambda^{(r)}[d_r = d] = \lambda[d])$$

with $\lambda^{(a)} \in \mathcal{A}_{k,n}^+$ and $0 \leq d_a - d_{a-1} \leq 1$ such that

$$\lambda^{(a)}/(d_a - d_{a-1})/\lambda^{(a-1)} := \{\langle i, j \rangle \in \mathbb{Z} \times \mathbb{Z} \mid \lambda_i^{(a)}[d_a] \geq j > \lambda_i^{(a-1)}[d_{a-1}]\}$$

⁴Compare with the “quantum Kostka number” introduced by Bertram, Ciocan-Fontanine, Fulton [6] in the context of the quantum cohomology ring of the Grassmannian.

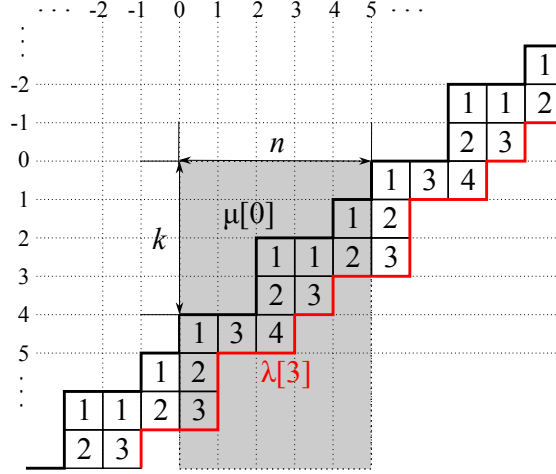


Figure 5.2: Example of a cylindric skew tableau. A set of representatives for the cylindric skew shape is shown in grey. Fix the (n, k) -bounding box as shown, then the upper boundary of the cylindric skew diagram cuts out the partition $\mu = (5, 4, 2, 2)$ while the lower one gives the partition $\Lambda = (5, 5, 5, 4, 3, 1, 1)$. Remove 3 rows of size $n = 5$ from the Young diagram of Λ to obtain the partition $\lambda = (4, 3, 1, 1)$ which now lies within the bounding box. Denote the cylindric skew diagram by $\lambda/d/\mu$. The depicted filling of the boxes with integers 1 to 4 yields an example of a cylindric tableau, the integers are weakly increasing in each row and strictly increasing in each column.

is a cylindric horizontal strip.

Proof. Given T define $\lambda^{(a)}/(d_a - d_{a-1})/\lambda^{(a-1)}$ to be the set of squares filled with a in T and let $\lambda^{(a)}[d_a], \lambda^{(a-1)}[d_{a-1}]$ be its lower and upper boundary. Because of (5.10) $\lambda^{(a)}/(d_a - d_{a-1})/\lambda^{(a-1)}$ must be a horizontal strip and, thus, we have $d_a - d_{a-1} = 0, 1$ otherwise there would be more than one box in the first column of the skew diagram in the fundamental region. We obviously have that $\lambda^{(0)} = \mu$ and $\lambda^{(r)} = \lambda$. The converse statement, that such a sequence of cylindric loops gives a cylindric tableau, is equally obvious. Requirement (5.9) is satisfied because each $\lambda^{(a)}/(d_a - d_{a-1})/\lambda^{(a-1)}$ is cylindric and, by definition, the degrees d_a of the individual loops accumulate to give d . (5.10) and (5.11) are satisfied since each cylindric sub-diagram $\lambda^{(a)}/(d_a - d_{a-1})/\lambda^{(a-1)}$ is horizontal. ■

5.3 Cylindric Hall-Littlewood functions

We now generalise the notion of skew Hall-Littlewood functions to cylindric skew diagrams and then link them to the partition function of the quasi-periodic q -boson model, analogous to Corollary 4.5.

We start with the generalisation of the coefficient functions (2.20) and (2.21) to cylindric skew tableaux. A cylindric horizontal strip of period $(k, -n)$ can obviously always be written as $\lambda/d/\mu$ with $\lambda, \mu \in \mathcal{A}_{k,n}^+$ and either $d = 0$ or $d = 1$ depending on the strip fitting within the (n, k) -bounding box or not. We introduce the following generalisations of the functions (2.20) and (2.21): let

$$\Phi_{\lambda/d/\mu}(t) = \begin{cases} \prod_{i \in I_{\lambda/d/\mu}} (1 - t^{m_i(\lambda)}), & \lambda/d/\mu \text{ is a horizontal strip} \\ 0, & \text{else} \end{cases} \quad (5.12)$$

and

$$\Psi_{\lambda/d/\mu}(t) = \begin{cases} \prod_{i \in J_{\lambda/d/\mu}} (1 - t^{m_i(\mu)}), & \lambda/d/\mu \text{ is a horizontal strip} \\ 0, & \text{else} \end{cases}. \quad (5.13)$$

Here the sets $I_{\lambda/d/\mu}$ and $J_{\lambda/d/\mu}$ are defined as follows: include $1 \leq i \leq n$ in $I_{\lambda/d/\mu}$ if $\lambda'[d]_i - \mu'[0]_i = 1$ and $\lambda'[d]_{i+1} - \mu'[0]_{i+1} = 0$, while $i \in J_{\lambda/d/\mu}$ if and only if $\lambda'[d]_i - \mu'[0]_i = 0$ and $\lambda'[d]_{i+1} - \mu'[0]_{i+1} = 1$.

Example 5.1 Set $n = 5, k = 6$ and $\ell = 3$. Consider the cylindric tableau shown on the right in Figure 5.3. Let us first compute the sets $I_{\lambda^{(i+1)}/d_i/\lambda^{(i)}}$ for the horizontal strips describing the first (top) skew tableau T_1 . This cylindric skew tableau corresponds to the sequence $(\mu[0] = \lambda^{(0)}[0], \lambda^{(1)}[d_1], \lambda^{(2)}[d_2], \lambda^{(3)}[d_3] = \lambda[d])$ with $\mu[0] = (\dots, 5, 5, 4, 3, 1, 1, \dots)$, $\lambda^{(1)}[0] = (\dots, 6, 5, 5, 3, 3, 1, \dots)$, $\lambda^{(2)}[1] = (\dots, 6, 5, 5, 4, 3, 2, 1, \dots)$ and $\lambda[d = 1] = (\dots, 6, 6, 5, 4, 4, 2, 1, \dots)$. We then find that $I_{\lambda^{(1)}/0/\mu[0]} = \{3, 6\}$, $I_{\lambda^{(2)}/1/\lambda^{(1)}} = \{4\}$ and $I_{\lambda^{(3)}/d/\lambda^{(2)}} = \{4, 6\}$.

Similarly, we find for the second (bottom) tableau T_2 in Figure 5.3 the sequence $(\lambda[0] = \mu^{(0)}[0], \mu^{(1)}[1], \mu^{(2)}[1], \mu^{(3)}[2] = \mu[2])$ with the cylindric loops $\mu^{(1)}[1] = (\dots, 6, 5, 5, 4, 3, 2, 1, \dots)$, $\mu^{(2)}[1] = (\dots, 6, 6, 5, 5, 3, 3, 1, \dots)$. Following the prescription outlined above we compute $J_{\mu^{(1)}/1/\lambda} = \{4, 6\}$, $J_{\mu^{(2)}/1/\mu^{(1)}} = \{4\}$ and $J_{\mu^{(3)}/2/\mu^{(2)}} = \{3, 6\}$. As the alert reader will have noticed these are the same sets as before but calculated in reverse order, since both results ought to reproduce the same Boltzmann weight $\Phi_{T_1}(t) = \Psi_{T_2}(t) = (1 - t)^4(1 - t^2)^2$ of the integrable q -boson lattice for the lattice configuration shown on the left in Figure 5.3; see Theorem 5.6.

Definition 5.5 (cylindric Hall-Littlewood functions) Given a (n, k) -restricted skew diagram $\lambda/d/\mu$ with $\lambda, \mu \in \mathcal{A}_{k,n}^+$ define the (restricted) cylindric skew Hall-Littlewood functions for $1 \leq \ell \leq k$ as

$$Q_{\lambda/d/\mu}(x_1, \dots, x_\ell; t) := \sum_{|T|=\lambda/d/\mu} \Phi_T(t) x^T, \quad (5.14)$$

and

$$P_{\lambda/d/\mu}(x_1, \dots, x_\ell; t) := \sum_{|T|=\lambda/d/\mu} \Psi_T(t) x^T, \quad (5.15)$$

where the sums run over all cylindric skew tableaux of shape $\lambda/d/\mu$ and Φ_T, Ψ_T are defined as the products of the functions (5.12) and (5.13) obtained when decomposing T into a sequence of horizontal strips. For $\ell > k$ we set both functions to zero.

Lemma 5.4 As in the non-cylindric case we have the identity

$$\Phi_{\lambda/d/\mu} = \frac{b_\lambda}{b_\mu} \Psi_{\lambda/d/\mu} \quad (5.16)$$

and, thus, $Q_{\lambda/d/\mu} = (b_\lambda/b_\mu)P_{\lambda/d/\mu}$; compare with (2.22).

Proof. Set $\theta_i = \lambda'[d]_i - \mu'[0]_i$ and observe that

$$m_i(\lambda) = \lambda'_i - \lambda'_{i+1} = \lambda'[d]_i - \lambda'[d]_{i+1},$$

and $\theta_i - \theta_{i+1} = m_i(\lambda) - m_i(\mu)$. Then

$$\begin{aligned} \frac{b_\lambda(t)}{b_\mu(t)} &= \frac{(t)_{m_1(\lambda)}(t)_{m_2(\lambda)} \cdots (t)_{m_n(\lambda)}}{(t)_{m_1(\mu)}(t)_{m_2(\mu)} \cdots (t)_{m_n(\mu)}} \\ &= \prod_{\theta_i - \theta_{i+1}=1} (1 - t^{m_i(\lambda)}) \prod_{\theta_i - \theta_{i+1}=-1} (1 - t^{m_i(\mu)})^{-1} = \Phi_{\lambda/d/\mu} / \Psi_{\lambda/d/\mu}, \end{aligned}$$

since $\theta_{n+1} = \lambda'[d]_{n+1} - \mu'[0]_{n+1} = \lambda'[d]_1 - \mu'[0]_1$. ■

With these definitions at hand we can now rewrite the action of the transfer matrix (3.42) in terms of cylindric horizontal strips.

Lemma 5.5 (cylindric horizontal strips) *We have the following alternative formulae for the action of the noncommutative elementary symmetric polynomials on the Fock space,*

$$e_r |\mu\rangle = \sum_{\substack{\lambda/d/\mu=(r), \\ \lambda \in \mathcal{A}_{k,n}^+}} z^d \Phi_{\lambda/d/\mu}(t) |\lambda\rangle = \sum_{\substack{\mu/d/\lambda=(n-r), \\ \lambda \in \mathcal{A}_{k,n}^+}} z^{1-d} \Psi_{\mu/d/\lambda}(t) |\lambda\rangle. \quad (5.17)$$

Note that the length of the horizontal strip in the fundamental region is different in both cases and that $d = 0$ or 1 .

Proof. To prove the assertion it suffices to show the identities $\psi_{\mu/\lambda} = \Phi_{\lambda/1/\mu}$, $\varphi_{\lambda/\mu} = \Psi_{\mu/1/\lambda}$ together with the fact that $\lambda/1/\mu$ being a horizontal r -strip implies that $\mu - \lambda = (n-r)$ and $\mu/1/\lambda$ being a horizontal $n-r$ strip is equivalent to $\lambda - \mu = (r)$. The last two claims are obvious from the definition of the cylindric skew diagram, since $\lambda[1]'_i - \mu_i[0]' = 1$ means that $\mu'_i - \lambda'_i = 0$ and vice versa. Recall that $\lambda[1]$ confined to the fundamental region is the Young diagram of λ with one row of length n added. Similarly, we have that $\mu[1]'_i - \lambda_i[0]' = 0, 1$ is equivalent to $\lambda'_i - \mu'_i = 1, 0$. The equality of the coefficient functions is now a trivial consequence of the definitions (2.20), (2.21) and (5.12), (5.13). ■

Theorem 5.6 *The cylindric skew HL functions are symmetric and we have the following identities with the quantum integrable q -boson model,*

$$Z_{\lambda/d/\mu}(x_1, \dots, x_\ell) = Q_{\lambda/d/\mu}(x_1, \dots, x_\ell; t) \quad (5.18)$$

$$= (x_1 \cdots x_\ell)^n P_{\mu/(\ell-d)/\lambda}(x_1^{-1}, \dots, x_\ell^{-1}; t), \quad (5.19)$$

where $1 \leq \ell \leq k$ as before.

Proof. We only need to prove the case $\ell = 1$, since according to Lemma 5.3, definitions (5.14), (5.15) and the fact that the weight of a lattice configuration equals the product of its row configurations the general case with $\ell \geq 1$ then trivially follows. We prove both assertions by formulating bijections between lattice configurations and cylindric tableau starting with (5.18).

Lattice-cylindric tableau bijection 1. Fix $\lambda, \mu \in \mathcal{A}_{k,n}^+$ and assume $\ell = 1$. Given an allowed row configuration $\sigma = (\sigma_1, \dots, \sigma_n, \sigma_1)$ we find from (4.6) that $\sigma_i = \lambda[\sigma_1]'_i - \mu[0]'_i = 0, 1$ for $1 \leq i \leq n$, where $\lambda[\sigma_1]'_i$, $\mu[0]'_i$ are the heights of the columns of the cylindric loops $\lambda[\sigma_1], \mu[0]$ in the fundamental region.

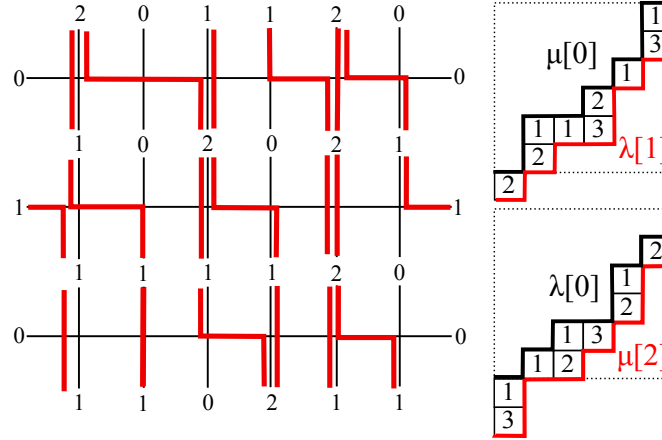


Figure 5.3: Example of a lattice configuration for the q -boson model with periodic boundary conditions. On the right the two cylindric skew tableaux obtained from bijection 1 (top) and bijection 2 (bottom) in the proof of Theorem 5.6. Here $\lambda = (6, 5, 4, 4, 2, 1)$ and $\mu = (5, 5, 4, 3, 1, 1)$.

Conversely, given a horizontal strip $\lambda/d/\mu$ we must have $d = 0$ or 1 , and, thus, we can define a unique row configuration via $\sigma_i := \lambda[\sigma_1]'_i - \mu[0]'_i$ with $\sigma_1 := \sigma_{n+1} := d$. Using the first equality of (5.17) in Lemma 5.5 the identity (5.18) follows.

For the benefit of the reader we describe the bijection between the set $\Gamma_{\lambda/d/\mu}$ of allowed lattice configurations (w.r.t. the vertex configurations shown in Figure (4.1)) and the set of cylindric skew tableaux of shape $\lambda/d/\mu$ also for the general case when $\ell \geq 1$. It is instructive to consult the example shown in Figure 5.3. Beginning at the first lattice row add a box with entry 1 in each column of the Young diagram of μ whenever the horizontal edge in the corresponding lattice column has value one. (This is analogous to the non-cylindric case discussed previously.) After arriving at the n th lattice column one obtains a finite horizontal strip, since in each column of the Young diagram at most one box is added. Continue the resulting horizontal strip from the fundamental region by shifting with the period vector $\Omega = (k, -n)$ to obtain the *cylindric* horizontal strip. Now do the same for the second lattice row labeling the boxes with 2 and so on. Continue until the last lattice row. The result is a uniquely defined cylindric skew tableau of shape $\lambda/d/\mu$. This bijection preserves weights and it generalises the one stated earlier in the context of the A -operator of the Yang-Baxter algebra.

Lattice-cylindric tableau bijection 2. For given $\lambda, \mu \in \mathcal{A}_{k,n}^+$ and $\ell = 1$ we now set $1 - \sigma_i = \mu[1 - \sigma_1]'_i - \lambda[0]'_i$ in order to identify a row configuration with a cylindric strip $\mu/d/\lambda$, where $d = 1 - \sigma_1$; compare with the second identity in (5.17) in Lemma 5.5. This proves (5.19) for $\ell = 1$ and the general case then follows by the same arguments as before.

Again for completeness we state the bijection also in the general case $\ell \geq 1$; it generalises the one for the D -operator from the previous section. Beginning with the bottom lattice row starting at $\langle 0, \ell \rangle$ place squares labelled with ℓ in each column of the extended Young diagram of $\mu[\ell - d]$, if the value of horizontal edge in the respective lattice column modulo n is zero. Continue to the next row and label the squares with $\ell - 1$

and so on. The result is a unique restricted skew cylindric tableau of shape $\mu/(\ell - d)/\lambda$. ■

Corollary 5.7 (Expansions of cylindric Hall-Littlewood functions) *Denote by $\mathcal{P}_{k,n}^+$ the set of partitions λ with $\ell(\lambda) \leq k$ and $\lambda_1 \leq n$. Then*

$$Q_{\lambda/d/\mu}(x; t) = \sum_{\nu \in \mathcal{P}_{k,n}^+} \langle \lambda | e_\nu | \mu \rangle m_\nu(x) = \sum_{\nu \in \mathcal{P}_{k,n}^+} \langle \lambda | s_{\nu'} | \mu \rangle s_\nu(x), \quad (5.20)$$

$$= \sum_{\nu \in \mathcal{P}_{k,n}^+} \langle \lambda | P'_{\nu'} | \mu \rangle P_\nu(x; t), \quad (5.21)$$

and

$$\langle \nu | e_\lambda | \mu \rangle = \sum_{\sigma \in \mathcal{P}_{k,n}^+} \langle \nu | s_{\sigma'} | \mu \rangle K_{\sigma\lambda} = \sum_{\substack{|T|=\nu/d/\mu \\ \text{wt}(T)=\lambda}} \Psi_T(t), \quad (5.22)$$

where $K = K(1)$ is the matrix of Kostka numbers and, hence,

$$\langle \nu | s_{\lambda'} | \mu \rangle = \sum_{w \in \mathfrak{S}_\ell} \varepsilon(w) \sum_{\substack{|T|=\nu/d/\mu \\ \text{wt } T=\lambda(w)}} \Psi_T(t)$$

with $\lambda(w) = (\lambda_1 - 1 + w_1, \dots, \lambda_\ell - \ell + w_\ell)$.

Proof. These expansion are an immediate consequence of the noncommutative Cauchy identities (3.55). ■

Remark 5.4 For $d = 0$ (5.22) becomes $\sum_T \psi_T(t) = b_\lambda(t)(K(t)^{-1}K)_{\lambda\mu}$ [53, III.6, eqn (6.4) on page 239]. Note that in general the expansion of $Q_{\lambda/d/\mu}(x; t)$ into Hall-Littlewood Q -functions does not yield polynomial coefficients in t , that is, $\langle \lambda | P'_{\nu'} | \mu \rangle / b_{\nu'}(t)$ is in general a rational function in t . In contrast, $\langle \lambda | P'_{\nu'} | \mu \rangle b_\mu(t) / b_\lambda(t)$ is always polynomial in t .

Example 5.2 Set $n = 4$ and $k = 3$. The following table summarises the expansion of the cylindric Hall-Littlewood function $P_{(3,2,1)/2/(4,3,1)}$ in various bases of the ring of symmetric functions.

λ	m_λ	s_λ	P_λ
4, 2, 0	$1 - t$	$1 - t$	$1 - t$
4, 1, 1	$2 - 3t + t^3$	$1 - 2t + t^3$	$1 - t - t^2 + t^3$
3, 3, 0	$2 - 3t + t^3$	$1 - 2t + t^3$	$1 - t - t^2 + t^3$
3, 2, 1	$6 - 14t + 5t^2 + 9t^3 - 7t^4 + t^5$	$2 - 8t + 5t^2 + 7t^3 - 7t^4 + t^5$	$2 - 5t + t^2 + 6t^3 - 5t^4 + t^5$
2, 2, 2	$10 - 26t + 14t^2 + 16t^3 - 21t^4 + 7t^5 + t^6 - t^7$	$1 - 3t + 4t^2 - 7t^4 + 5t^5 + t^6 - t^7$	$1 - t - t^2 - t^3 + t^4 + 4t^5 - 3t^6$

In comparison we have the following expansion coefficients for the ordinary skew Hall-Littlewood function $P_{(4,4,3,2,1)/(4,3,1,0,0)}$:

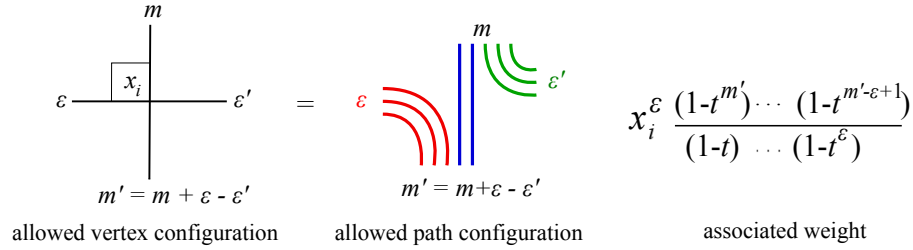


Figure 6.1: The vertex configurations and weights of the statistical model associated with the second solution to the Yang-Baxter equation (6.1).

λ	m_λ	s_λ	P_λ
4, 2, 0	1	1	1
4, 1, 1	$2 - t - t^2$	$1 - t - t^2$	$1 - t^2$
3, 3, 0	$2 - t - t^2$	$1 - t - t^2$	$1 - t^2$
3, 2, 1	$7 - 7t - 4t^2 + 4t^3$	$3 - 5t - 2t^2 + 4t^3$	$3 - 2t - 3t^2 + 2t^3$
2, 2, 2	$12 - 15t - 6t^2 + 11t^3 - t^4 - t^5$	$1 - 3t + 3t^3 - t^4 - t^5$	$1 - t^2 - t^3 + t^5$

Proposition 5.8 (inverse Kostka-Foulkes matrix) Denote by n^k the partition whose Young diagram consists of k rows of size n . Let $\lambda \in \mathcal{A}_{k,n}^+$ and $\tilde{\lambda}$ be the partition with all n -parts removed. Then

$$Q_{\lambda/d/n^k}(x_1, \dots, x_k; t) = Q_{\tilde{\lambda}}(x_1, \dots, x_k; t) \quad (5.23)$$

with $d = k - m_n(\lambda)$ and in particular we have that

$$\langle \lambda | \mathbf{s}_{\mu'} | n^k \rangle = b_{\tilde{\lambda}}(t) K(t)_{\tilde{\lambda}\mu}^{-1}. \quad (5.24)$$

Proof. Exploit the description of lattice configurations in terms of non-intersecting paths using the correspondence shown in Figure 4.1 between vertex and paths configurations. According to our assumptions there are $d = \sum_{i=1}^{n-1} m_i(\lambda)$ paths crossing the boundary, none of which ends up in the n th lattice column. Therefore, we must have for each cylindric tableau T of shape $\lambda/d/n^k$ that $\Phi_T = \varphi_{\tilde{T}}$ with \tilde{T} being the ordinary tableau of shape $\tilde{\lambda}$ obtained by restricting T to the fundamental region. Thus, the first assertion follows. The second is then a simple consequence of the definition (3.51) and the known transformation matrix $M(Q, s) = b(t)K(t)^{-1}$; see [53, III.6, Table on p241]. ■

6 Cylindric Macdonald functions

In this section we define cylindric analogues of the skew Macdonald functions $Q'_{\lambda/\mu}$ and $P'_{\lambda/\mu}$; see (2.26) for their definition.

6.1 The statistical vertex model associated with L'

We consider again a statistical vertex model defined on a cylinder but this time we assume that the number of lattice rows of \mathbb{L} lies in the interval $1 \leq \ell \leq n-1$. The setup and definitions remain the same as previously with the following two exceptions:

1. A horizontal edge configuration is now a map $\gamma'_h : \mathbb{E}_h \rightarrow \mathbb{Z}_{\geq 0}$, that is, vertical *and* horizontal edges can take values in the nonnegative integers. We shall label values of horizontal edge with the Greek letter ε and continue to label vertical ones with the letter m . With this change a vertex configuration $\gamma'_{\langle i, j \rangle}$ is now a 4-tuple of nonnegative integers, $\gamma'_{\langle i, j \rangle} = \{\varepsilon, m, \varepsilon', m'\}$ giving the values of the W, N, E and S edges centered at the interior lattice point $\langle i, j \rangle$, respectively; see Figure 6.1
2. We now fix the weights of the vertex configurations through the matrix elements of the L' -operator (3.20),

$$\text{wt}'(\gamma'_{\langle i, j \rangle}) := \langle \varepsilon', m' | L'(x_i) | \varepsilon, m \rangle = u^\varepsilon \begin{bmatrix} m' \\ \varepsilon \end{bmatrix}_t \delta_{m+\varepsilon, m'+\varepsilon'} . \quad (6.1)$$

As previously we then define the weight of a lattice configuration $\gamma' = (\gamma'_h, \gamma'_v)$ as the product of vertex configurations over interior lattice points,

$$\text{wt}'(\gamma') = \prod_{\langle i, j \rangle \in \tilde{\mathbb{L}}} \text{wt}'(\gamma'_{\langle i, j \rangle}) \quad (6.2)$$

and call a vertex or lattice configuration “not allowed” if the corresponding weight vanishes.

Remark 6.1 *The vertex configurations can again be interpreted in terms of non-intersecting paths or “infinitely-friendly walkers”; see Figure 6.1. Note, however, that this particular statistical model has not been previously discussed in the literature.*

Shadowing closely our previous discussion of the q -boson model we consider again for $\lambda, \mu \in \mathcal{A}_{k,n}^+$ the set $\Gamma'_{\lambda, \mu}$ of periodic or cylindric lattice configurations, that is $\gamma'_h(e_{i,0}) = \gamma'_h(e_{i,n})$ for each $\gamma' \in \Gamma'_{\lambda, \mu}$ where $e_{i,0}$ and $e_{i,n}$ are the horizontal edges starting at the points $\langle i, 0 \rangle$ and $\langle i, n \rangle$, respectively. The values of the top and bottom outer vertical edges are fixed in terms of the multiplicities $m_j(\mu)$ and $m_j(\lambda)$ as previously discussed and we set similar as before

$$\Gamma'_{\lambda/d/\mu} := \{ \gamma' \in \Gamma'_{\lambda, \mu} : \sum_{i=1}^{\ell} \gamma'_h(\langle i, 0 \rangle, \langle i, 1 \rangle) = d \} .$$

That is, $\Gamma'_{\lambda/d/\mu}$ is the subset of configurations where the sum over the values of the left (or right) outer horizontal edges is d . Define the partition function $Z'_{\lambda/d/\mu}(x_1, \dots, x_\ell) := \sum_{\gamma' \in \Gamma'_{\lambda/d/\mu}} \text{wt}'(\gamma')$ then we have in close analogy to our previous discussion the following identity.

Lemma 6.1 *Let $\lambda, \mu \in \mathcal{A}_{k,n}^+$. The following equality between matrix element and partition functions is true,*

$$\langle \lambda | \mathbf{G}'(x_1) \cdots \mathbf{G}'(x_\ell) | \mu \rangle = \sum_{d \geq 0} z^d Z'_{\lambda/d/\mu}(x_1, \dots, x_\ell) . \quad (6.3)$$

Moreover, $Z'_{\lambda/d/\mu}(x_1, \dots, x_\ell)$ is symmetric in the x_i 's.

Proof. The assertion is again a direct consequence of the definitions (3.46) and (6.1). That the partition function is symmetric follows from $\mathbf{G}'(x_i)\mathbf{G}'(x_j) = \mathbf{G}'(x_j)\mathbf{G}'(x_i)$ which is a consequence of (3.22). ■

Note that we can recover “open boundary conditions”, that is configurations on a finite strip with the values of the outer horizontal edges all being zero, by setting $z = 0$.

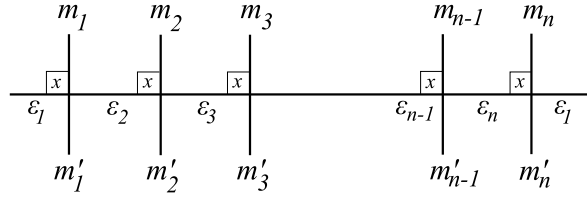


Figure 6.2: The numbering convention used in describing a row configuration of the vertex model (6.1).

6.2 Cylindric weight functions

We now introduce the cylindric analogue of the coefficient function (2.25) which assigns a weight to each cylindric horizontal strip of the form $\lambda'/d/\mu'$. Here $\lambda'/d/\mu'$ is the conjugate or transposed cylindric skew diagram of $\lambda/d/\mu$ defined earlier, that is we now consider the case when $\lambda/d/\mu$ is a *vertical* strip. For $\lambda, \mu \in \mathcal{A}_{k,n}^+$ set

$$\Phi'_{\lambda'/d/\mu'}(t) := \prod_{j=1}^n \left[\lambda'[d]_{j+1} - \mu'[0]_{j+1} \right]_t \quad (6.4)$$

if $\lambda'/d/\mu'$ is a horizontal strip and zero otherwise. To facilitate the comparison with (2.25) note that we have

$$\lambda'[d]_1 - \mu'[0]_1 = k + d - k = d - 0 = \lambda'[d]_{n+1} - \mu'[0]_{n+1} \quad (6.5)$$

and, hence, the analogue of the additional factor $1/(t)_{\lambda'_1 - \mu'_1}$ appearing in (2.25) is included in the definition (6.4). Thus, the polynomial $\Phi'_{\lambda'/d/\mu'}$ can be interpreted as the natural generalisation of $\varphi'_{\lambda'/\mu'}$ to cylindric skew diagrams. We also introduce the cylindric counterpart for the second coefficient function (2.24) setting

$$\Psi'_{\lambda'/d/\mu'}(t) = \prod_{j=1}^n \left[\lambda'_j - \lambda'_{j+1} \right]_t. \quad (6.6)$$

The following equality shows that the two definitions (6.4) and (6.6) are consistent with each other.

Lemma 6.2 *We have the identity*

$$\Psi'_{\lambda'/d/\mu'}(t) = \frac{b_\lambda(t)}{b_\mu(t)} \Phi'_{\lambda'/d/\mu'}(t). \quad (6.7)$$

Proof. A trivial rewriting using that $\lambda'_j - \lambda'_{j+1} = \lambda'[0]_j - \lambda'[0]_{j+1} = \lambda'[d]_j - \lambda'[d]_{j+1}$. ■

Definition 6.1 *Let $\lambda, \mu \in \mathcal{A}_{k,n}^+$ and $d \geq 0$, $1 \leq \ell \leq n-1$. Then we define the (restricted) cylindric Macdonald functions $P'_{\lambda'/d/\mu'}$ and $Q'_{\lambda'/d/\mu'}$ as follows,*

$$P'_{\lambda'/d/\mu'}(x_1, \dots, x_\ell; t) := \sum_{|T|=\lambda'/d/\mu'} \Psi'_T(t) x^T, \quad (6.8)$$

and

$$Q'_{\lambda'/d/\mu'}(x_1, \dots, x_\ell; t) := \frac{b_\mu(t)}{b_\lambda(t)} P'_{\lambda'/d/\mu'}(x_1, \dots, x_\ell; t),$$

where the sum runs over all conjugate cylindric skew tableaux T of shape $\lambda'/d/\mu'$.

The following proposition ties the transfer matrix (3.46) of the second statistical model to cylindric vertical strips $\lambda/d/\mu$.

Proposition 6.3 (cylindric vertical strips) *Let $\mathbf{G}'(u) := \sum_{r \geq 0} u^r \mathbf{g}'_r$ be the formal partial trace of the monodromy introduced above. The latter is well-defined as an operator in $\text{End}(\mathcal{F}^{\otimes n})$ since when acting on $|\mu\rangle$, $\mu \in \mathcal{A}_{k,n}^+$ for any $k \geq 0$ only a finite number of coefficients act non-trivially. Namely, one has*

$$\mathbf{g}'_r |\mu\rangle = \sum_{\substack{\lambda/d/\mu=(1^r), \\ \lambda \in \mathcal{A}_{k,n}^+}} z^d \Psi'_{\lambda'/d/\mu'}(t) |\lambda\rangle, \quad (6.9)$$

where the second sum runs over all cylindric vertical strips of length r with $0 \leq d \leq \min(r, m_n(\mu))$.

Remark 6.2 *The operators \mathbf{g}'_r are closely related to the discrete Laplacians considered in [73] and the functional relation (3.50) can be seen as a generalisation of Baxter's famous TQ-relation for the six-vertex or XXZ model with \mathbf{E} corresponding to the transfer matrix T and \mathbf{G}' to Baxter's Q -operator. In fact, (3.46) is a generalisation of the XXZ Q -operator construction given in [43] for "infinite spin".*

Proof. Let $\mu \in \mathcal{A}_{k,n}^+$. According to the definitions (3.46) and (3.47) the matrix elements of \mathbf{g}'_r give the sum over all allowed row configurations. Fix an allowed horizontal edge configuration $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n, \varepsilon_{n+1}) \in \mathbb{Z}_{\geq 0}^{n+1}$, that is $m_j(\mu) \geq \varepsilon_{j+1}$ and $\varepsilon_{n+1} = \varepsilon_1$; see Figure 6.2 for an illustration and note that we must have in particular $\varepsilon_1 \leq m_n$. Denote by λ the partition in $\mathcal{A}_{k,n}^+$ corresponding to the resulting multiplicities $m'_j := m_j + \varepsilon_j - \varepsilon_{j+1}$ and by $\lambda'[\varepsilon_1]$ the associated conjugate cylindric loop with $\lambda'[\varepsilon_1]_n = m'_n + \varepsilon_1 = m_n + \varepsilon_n$. Then we have the following relations between conjugate cylindric loops and horizontal edge values,

$$m'_j = \lambda'[\varepsilon_1]_j - \lambda'[\varepsilon_1]_{j+1} \quad \text{and} \quad \varepsilon_j = \lambda'[\varepsilon_1]_j - \mu'[0]_j, \quad j = 1, \dots, n. \quad (6.10)$$

Furthermore, it follows from $\varepsilon_1 \leq m_n$ that the conjugate cylindric skew diagram $\lambda'/\varepsilon_1/\mu'$ is a horizontal strip of length $r = \sum_{j=1}^n \varepsilon_j$ in the fundamental region, where ε_j boxes are added in the j th row of the Young diagram of μ' . The associated weight of this row configuration is according to (6.1) given by

$$x_i^{\varepsilon_1 + \dots + \varepsilon_n} \prod_{j=1}^n \left[\frac{\lambda'_j - \lambda'_{j+1}}{\lambda'[\varepsilon_1]_j - \mu'[0]_j} \right]_t = x_i^{\varepsilon_1 + \dots + \varepsilon_n} \Psi'_{\lambda'/\varepsilon_1/\mu'} \quad (6.11)$$

which for configurations with $\varepsilon_1 = 0$ coincides with $x_i^{\varepsilon_2 + \dots + \varepsilon_n} \psi'_{\lambda'/\mu'}(t)$; see the definition in (2.24). Conversely, given a conjugate cylindric diagram $\lambda'/d/\mu'$ which is a horizontal strip the lattice row configuration can be reconstructed in a unique way using the formulae in (6.10) with $\varepsilon_1 = d$. ■

In analogy with Theorem 5.6 we now have the following result:

Theorem 6.4 *Let $\lambda, \mu \in \mathcal{A}_{k,n}^+$. The cylindric skew Macdonald functions $P'_{\lambda'/d/\mu'}$, $Q'_{\lambda'/d/\mu'}$ are symmetric in the variables $x = (x_1, \dots, x_\ell)$ and one has the expansion*

$$\langle \lambda | \mathbf{G}'(x_1) \cdots \mathbf{G}'(x_\ell) | \mu \rangle = \sum_{d \geq 0} z^d P'_{\lambda'/d/\mu'}(x_1, \dots, x_\ell; t) \quad (6.12)$$

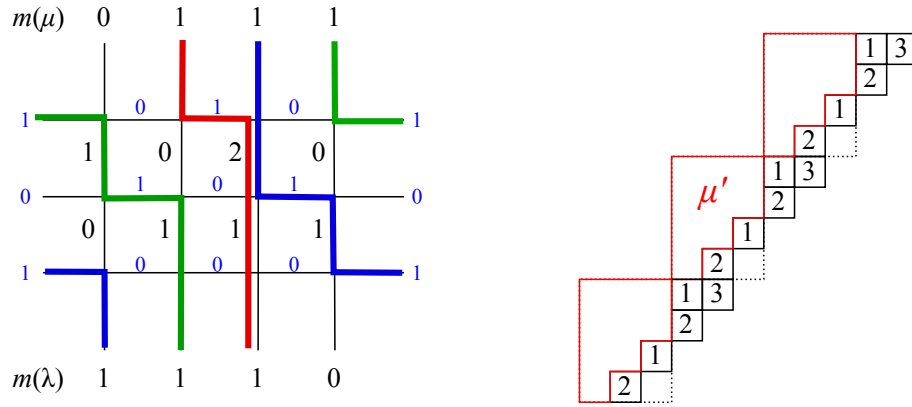


Figure 6.3: Example for the bijection between the lattice configurations of the vertex model (6.1) and cylindric skew tableaux.

which is equivalent to

$$Z'_{\lambda/d/\mu}(x_1, \dots, x_\ell) = P'_{\lambda'/d/\mu'}(x_1, \dots, x_\ell; t). \quad (6.13)$$

Here $1 \leq \ell \leq n-1$ as before.

Remark 6.3 Setting $z = 0$ the above result specialises to the identity

$$Z'_{\lambda/0/\mu}(x_1, \dots, x_\ell) = P'_{\lambda'/\mu'}(x_1, \dots, x_\ell; t), \quad (6.14)$$

where $P'_{\lambda'/\mu'}$ is the ordinary (non-cylindric) Macdonald function defined in (2.26). In particular, $Z'_{\lambda/0/\mu}(x_1, \dots, x_\ell) = 0$ unless $\mu \subset \lambda$.

Proof. Set $\ell = 1$ then the statement is an immediate consequence of the previous proposition. The case $\ell > 1$ now trivially follows, since each conjugate cylindric tableau $\lambda'/d/\mu'$ can be written as a sequence over (conjugate) cylindric horizontal strips: given an allowed configuration γ' of the entire lattice, denote by $m^{(i)} = (m_1^{(i)}, \dots, m_n^{(i)})$ the upper vertical, and by $\varepsilon^{(i)} = (\varepsilon_1^{(i)}, \dots, \varepsilon_n^{(i)})$ the horizontal edge values in the i^{th} lattice row. Set $m^{(0)} = m(\mu)$, $m^{(\ell)} = m(\lambda)$ and denote by $\lambda^{(i)}$ the partition whose Young diagram has $m_j^{(i)}$ columns of length j . Then $\lambda^{(i)}[\sum_{l=1}^i \varepsilon_1^{(l)}]$ is the associated conjugate cylindric loop, that is in the fundamental region we add to the partition $\lambda^{(i)}$ the total of $\sum_{l=1}^i \varepsilon_1^{(l)}$ columns of height n . Thus, each row adds a horizontal strip of length $r_i = |\varepsilon^{(i)}|$ to the Young diagram of μ in the fundamental region. The sequence

$$(\lambda^{(0)}[0], \lambda^{(1)}[\varepsilon_1^{(1)}], \lambda^{(2)}[\varepsilon_1^{(1)} + \varepsilon_1^{(2)}], \dots, \lambda^{(\ell-1)}[\varepsilon_1^{(1)} + \dots + \varepsilon_1^{(\ell-1)}], \lambda^{(\ell)}[d])$$

with $\mu'[0] = \lambda^{(0)}[0]$, $\lambda^{(\ell)}[d] = \lambda'[d]$, $d = \sum_{i=1}^{\ell} \varepsilon_1^{(i)}$ results in a unique conjugate cylindric skew tableaux T of shape $\lambda'/d/\mu'$, because in each row we have the constraint $\varepsilon_1^{(i+1)} \leq m_n^{(i)}$ and (6.10) must hold true for an allowed configuration. Moreover, the weight of the tableau is given as product over the row weights, see (6.2), and we recall that $\Psi'_T := \prod_{i=0}^{n-1} \Psi'_{\lambda^{(i+1)}/\varepsilon_1^{(i+1)}/\lambda^{(i)}}$. ■

Corollary 6.5 (Expansions of cylindric Macdonald functions) *We have the following expansions of the cylindric functions (6.8)*

$$P'_{\lambda'/d/\mu'}(x_1, \dots, x_{n-1}; t) = \sum_{\nu \in \tilde{\mathcal{A}}_{k,n}^+} \langle \lambda | \mathbf{g}'_{\nu'} | \mu \rangle m_{\nu'}(x_1, \dots, x_{n-1}) \quad (6.15)$$

$$= \sum_{\nu \in \tilde{\mathcal{A}}_{k,n}^+} \langle \lambda | \mathbf{S}'_{\nu'} | \mu \rangle s_{\nu'}(x_1, \dots, x_{n-1}), \quad (6.16)$$

$$= \sum_{\nu \in \tilde{\mathcal{A}}_{k,n}^+} \langle \lambda | \mathbf{Q}'_{\nu'} | \mu \rangle P'_{\nu'}(x_1, \dots, x_{n-1}; t), \quad (6.17)$$

where m are the monomial symmetric functions, s the Schur functions and P' the Macdonald functions defined in (2.26). The expansion coefficients obey the relations

$$\langle \lambda | \mathbf{g}'_{\nu'} | \mu \rangle = \sum_{\sigma \in \tilde{\mathcal{A}}_{k,n}^+} \langle \lambda | \mathbf{S}'_{\sigma'} | \mu \rangle K_{\sigma\nu} = \sum_{\substack{|T|=\lambda'/d/\mu' \\ \text{wt}(T)=\nu'}} \Psi'_T(t) \quad (6.18)$$

with $K = M(s, m)$ being the Kostka-matrix [53] and

$$\langle \lambda | \mathbf{S}'_{\nu'} | \mu \rangle = \sum_{\sigma \in \tilde{\mathcal{A}}_{k,n}^+} K_{\nu\sigma}(t) \langle \lambda | \mathbf{Q}'_{\sigma'} | \mu \rangle = \sum_{w \in \mathfrak{S}_\ell} \varepsilon(w) \sum_{\substack{|T|=\lambda'/d/\mu' \\ \text{wt } T=\nu'(w)}} \Psi'_T(t), \quad (6.19)$$

where $K(t) = M(s, P)$, $\ell = \ell(\nu')$ and $\nu'(w) := (\nu'_1 - 1 + w_1, \dots, \nu'_\ell - \ell + w_\ell)$.

Proof. The first three equalities, (6.15), (6.16) and (6.17) are direct consequences of the noncommutative Cauchy expansions (3.56). Exploiting the known transition matrix from the basis of Schur functions to the basis of monomial symmetric functions the asserted relation (6.18) now follows from (6.13). The last relation, Equation (6.19), is then a direct consequence of the definition (3.52) of $\mathbf{S}'_{\nu'}$,

$$\langle \lambda | \mathbf{S}'_{\nu'} | \mu \rangle = \sum_{w \in \mathfrak{S}_\ell} \varepsilon(w) \langle \lambda | \mathbf{g}'_{\nu'_\ell - \ell + w_\ell} \cdots \mathbf{g}'_{\nu'_1 - 1 + w_1} | \mu \rangle,$$

and the fact that $P'_{\lambda'} = \sum_{\mu} s_{\mu'} K_{\mu\lambda}(t)$; see [53]. ■

Example 6.1 *Let $n = k = 5$ and consider $P'_{(5,3,2,1,1)/1/(5,3,1,0,0)}$. Then we find the expansion coefficients in the m and s -bases displayed in the table below:*

λ	m_λ	s_λ
4, 2, 2, 0	$1 + t$	$1 + t$
4, 2, 1, 1	$2 + 3t + t^2$	$1 + 2t + t^2$
3, 3, 2, 0	$2 + 3t + t^2$	$1 + 2t + t^2$
3, 3, 1, 1	$4 + 8t + 5t^2 + t^3$	$1 + 3t + 3t^2 + t^3$
3, 2, 2, 1	$11 + 22t + 16t^2 + 4t^3$	$3 + 8t + 9t^2 + 3t^3$
2, 2, 2, 2	$24 + 52t + 45t^2 + 16t^3 + t^4$	$1 + 4t + 6t^2 + 5t^3 + t^4$

N.B. in these expansions, as in the definition of $P'_{\lambda'/d/\mu'}$, we have assumed the number of variables to be at most $n - 1$.

6.3 Kostka-Foulkes polynomials

As a special case one can also recover the non-skew P', Q' -functions from the cylindric functions.

Proposition 6.6 (Kostka-Foulkes polynomials) *Let $\lambda \in \mathcal{A}_{k,n}^+$ and $\tilde{\lambda}$ the partition with all parts of size n removed. Set $d = k - m_n(\lambda)$. Then*

$$P'_{\lambda'/d/k^n}(x; t) = P'_{\tilde{\lambda}}(x; t) , \quad (6.20)$$

where k^n is the partition with n parts of size k . In particular, for any $\nu \in \tilde{\mathcal{A}}_{k,n}^+$ we have the following equalities

$$\langle \lambda | g'_{\nu'} | n^k \rangle = \sum_{\mu} K_{\mu \tilde{\lambda}}(t) K_{\mu' \nu'} = \sum_{\substack{|T|=\tilde{\lambda}' \\ \text{wt}(T)=\nu'}} \psi'_T(t) \quad (6.21)$$

and

$$K_{\nu \tilde{\lambda}}(t) = \langle \lambda | S'_{\nu'} | n^k \rangle = \sum_w \varepsilon(w) \sum_{\substack{|T|=\tilde{\lambda}' \\ \text{wt } T=\nu'(w)}} \psi'_T(t) , \quad (6.22)$$

where $K_{\nu \tilde{\lambda}}(t)$ is the celebrated Kostka-Foulkes polynomial and $\nu'(w) = (\nu'_1 - 1 + w_1, \dots, \nu'_\ell - \ell + w_\ell)$.

Proof. Because of (6.13) we have

$$P'_{\lambda'/d/k^n}(x) = \frac{d^d}{dz^d} \langle \lambda | \mathbf{G}'(x_1) \cdots \mathbf{G}'(x_\ell) | n^k \rangle \Big|_{z=0} .$$

We use that each lattice configuration corresponds to a configuration of non-intersecting paths on the lattice; see Figure 6.1. All paths start at the rightmost, the n th, lattice column and there must be d paths crossing the boundary. None of these paths ends up in the n th lattice column, since $\sum_{i=1}^{n-1} m_i(\lambda) = d$ (and the paths cannot backtrack on themselves), hence one easily verifies that $\Psi'_T = \psi'_{\tilde{T}}$. Here T is the cylindric tableau of shape $\lambda'/d/k^n$ and \tilde{T} the tableau of shape $\tilde{\lambda}'$ which is obtained by only considering the squares of $\lambda'/d/k^n$ in the fundamental region. It is now obvious from (6.6) and (2.24) that $\text{wt}(T) = \text{wt}(\tilde{T})$. This proves the first assertion.

Employing the known expansion $P'_{\nu'} = \sum_{\lambda} s_{\lambda'} K_{\lambda \nu}(t)$ where $K_{\lambda \nu}(t)$ are the (ordinary) Kostka-Foulkes polynomials [53], it follows from our earlier result that

$$P'_{\lambda'/d/k^n}(x; t) = \sum_{\lambda} s_{\lambda'}(x) K_{\lambda \tilde{\nu}}(t) .$$

Comparing this with (6.18) and (6.19) proves the remaining assertions. ■

The expression (6.22) for Kostka-Foulkes polynomials is of determinant type and not manifestly positive unlike other expressions [48] [36], see also [53, III.6, eqn (6.5) on p242 and Ex. 7 on p245]. The present formula resembles instead Lusztig's t -deformed weight multiplicity formula [50]

$$K_{\lambda \mu}(t) = \sum_{w \in \mathfrak{S}_k} (-1)^{|w|} \mathcal{P}_t(w(\lambda + \rho) - (\mu + \rho)),$$

where $\rho = (k, k-1, \dots, 1)$ is the Weyl vector and the weight multiplicity is given through the t -analogue of Konstant's partition function

$$\prod_{\alpha > 0} \frac{1}{1 - te^\alpha} = \sum_{\mu \in \mathbb{Z}^k} \mathcal{P}_t(\mu) e^\mu.$$

That is, the coefficient of t^m in $\mathcal{P}_t(\mu)$ is the number of ways the weight μ can be expressed as a sum of m positive roots.

Example 6.2 Set $\lambda = (3, 3, 2, 0)$ and $\mu = (2, 2, 2, 2)$. Then we find for $K_{\lambda\mu}(t)$ the following non-vanishing summands employing formula (6.22),

$\varepsilon(w)$	w	$\lambda'(w)$	ψ'_T
+	123	(3, 3, 2)	$1 + 2t + 3t^2 + 3t^3 + 2t^4 + t^5$
−	213	(4, 2, 2)	$1 + t + 2t^2 + t^3 + t^4$
−	132	(4, 3, 1)	$1 + t + t^2 + t^3$
+	231	(4, 4, 0)	1

and, hence, that $K_{\lambda\mu}(t) = t^3 + t^4 + t^5$. In comparison, using the t -analogue of Konstant's partition function one arrives at (compare e.g. with [16, Example 3.1])

$\varepsilon(w)$	$w(\lambda + \rho)$	$w(\lambda + \rho) - (\mu + \rho)$	$\mathcal{P}_t(w(\lambda + \rho) - (\mu + \rho))$
+	(6, 5, 3, 0)	(1, 1, 0, −2)	$t^2 + 3t^3 + 2t^4 + t^5$
+	(5, 6, 3, 0)	(0, 2, 0, −2)	$t^2 + t^3 + t^4$
−	(6, 3, 5, 0)	(1, −1, 2, −2)	t^3

reproducing the same result for the Kostka-Foulkes polynomial.

7 A deformation of the Verlinde algebra

In this section we will identify the expansion coefficients of the cylindric Macdonald functions in Corollary 6.5, i.e. the matrix elements $\langle \nu | Q'_{\lambda'} | \mu \rangle$ and $\langle \nu | S'_{\lambda'} | \mu \rangle$, with the *structure constants* of an algebra which is a quotient of the spherical Hecke algebra. More precisely, this quotient is the coordinate ring $\mathbb{k}[\mathbf{V}_{k,n}]$ of a discrete (0-dimensional) affine variety $\mathbf{V}_{k,n}$ where \mathbb{k} is the field of Puiseux series in the indeterminate $t = q^2$ with complex coefficients. In particular, $\mathbb{k}[\mathbf{V}_{k,n}]$ is finite-dimensional and a commutative Frobenius algebra $\mathfrak{F}_{n,k}$. The latter has a distinguished basis which we can identify with $\{Q'_{\lambda'} : \lambda \in \mathcal{A}_{k,n}^+\}$ and the coproduct $\Delta_{n,k} : \mathfrak{F}_{n,k} \rightarrow \mathfrak{F}_{n,k} \otimes \mathfrak{F}_{n,k}$ of the Frobenius algebra computed in this basis yields the cylindric Macdonald function (6.8), $\Delta_{n,k} P'_{\lambda'} = \sum_{d, \mu'} P'_{\lambda'/d/\mu'} \otimes P'_{\mu'}$.

We arrive at this result by constructing a common eigenbasis for the noncommutative Macdonald functions (3.51) and (3.52). We apply a technique known as algebraic Bethe ansatz which we discuss in the next subsection. Each point in the affine variety $\mathbf{V}_{k,n}$ will determine an eigenvector and the latter corresponds to an idempotent of $\mathfrak{F}_{n,k}$. This correspondence will allow us to identify $\mathfrak{F}_{n,k}$ with the subalgebra $\subset \text{End } \mathcal{F}_k^{\otimes n}$ generated by the matrices $Q_{\nu'}^{(k)} := (\langle \lambda | Q'_{\nu'} | \mu \rangle)_{\lambda, \mu \in \mathcal{A}_{k,n}^+}$. The latter specialise for $t = 0$ to the fusion matrices of the Verlinde algebra and, hence, we will refer to $\mathfrak{F}_{n,k}$ as *deformed* fusion ring or Verlinde algebra of $\widehat{\mathfrak{sl}}(n)_k$.

7.1 The Bethe ansatz

The Bethe ansatz is a well-established technique in the physics literature on exactly-solvable lattice and quantum integrable models [4]. Here we will employ the so-called *algebraic* Bethe ansatz also known as the *quantum inverse scattering method* (see e.g. [8] and references therein) which uses the commutation relations of the Yang-Baxter algebra (3.18) to construct eigenvectors, so-called “Bethe vectors”, of the q -boson model transfer matrix (3.42); compare with the discussion in [7]. Using Proposition 3.50 one deduces that these are also eigenvectors of the formal power series (3.46). This allows one to recover the results in [73] where the eigenvectors of certain “discrete Laplacians” for the quantum nonlinear Schrödinger have been computed using the so-called *coordinate* Bethe ansatz.

Our starting point is the *ansatz* that the eigenvectors of (3.42) are of the algebraic form

$$\mathbf{b}(y) := B(y_1^{-1}) \cdots B(y_k^{-1})|\emptyset\rangle = \sum_{\lambda \in \mathcal{A}_{k,n}^+} Q_\lambda(y^{-1}; t) |\lambda\rangle, \quad (7.1)$$

where $y = (y_1, \dots, y_k)$ are some *dependent* invertible variables, called *Bethe roots*, which need to be determined. We specify them below as points in a family of affine varieties depending on t , for the moment we treat them as formal variables. Using the commutation relations (3.29) and (3.30) of the Yang-Baxter algebra, one arrives at a set of nonlinear equations, called the *Bethe ansatz equations* and an expression of the eigenvalue for the transfer matrix (3.42).

Proposition 7.1 (Bogoliubov-Izergin-Kitanine) (i) *The Bethe vector $\mathbf{b}(y)$ is an eigenvector of the transfer matrix \mathbf{E} provided the Bethe roots $y = (y_1, \dots, y_k)$ satisfy the following set of coupled nonlinear equations,*

$$y_i^n \prod_{j \neq i} \frac{y_i - y_j t}{y_i - y_j} = z \prod_{j \neq i} \frac{y_i t - y_j}{y_i - y_j}, \quad i = 1, \dots, k. \quad (7.2)$$

(ii) *Let y be a solution to the Bethe ansatz equations (7.2). Then one has the eigenvalue equation*

$$\mathbf{E}(u) \mathbf{b}(y) = \left(\prod_{i=1}^k \frac{1 - u y_i t}{1 - u y_i} + z t^k u^n \prod_{i=1}^k \frac{1 - u t^{-1} y_i}{1 - u y_i} \right) \mathbf{b}(y). \quad (7.3)$$

Proof. The asserted identities are derived via induction in k exploiting the commutation relations (3.28), (3.29), (3.30) and (3.31) of the Yang-Baxter algebra. This is a standard computation (see [8] for a textbook reference) and we therefore omit it. ■

The following result is not contained in [7].

Corollary 7.2 *Under the same set of conditions (7.2) as above the Bethe vector $\mathbf{b}(y)$ is also an eigenvector of the transfer matrix \mathbf{G}' ,*

$$\mathbf{G}'(u) \mathbf{b}(y) = \prod_{i=1}^k (1 + u y_i) \mathbf{b}(y). \quad (7.4)$$

This, in particular implies that

$$\mathbf{Q}'_{\lambda'} \mathbf{b}(y) = P_\lambda(y; t) \mathbf{b}(y), \quad (7.5)$$

where $\mathbf{Q}'_{\lambda'}$ is the noncommutative HL polynomial defined in (3.52).

Proof. The assertion is a direct consequence of the identity (3.50) we proved earlier. Employing the expansion (3.54) one verifies that \mathbf{g}'_r can be written as a polynomial in the \mathbf{e}_i 's. For instance, we have $(1-t)\mathbf{g}'_1 = \mathbf{e}_1$, $(1-t^2)\mathbf{g}'_2 = (1-t)^{-1}\mathbf{e}_1 - \mathbf{e}_2$ etc. Hence, $\mathbf{b}(y)$ must be an eigenvector of $\mathbf{G}'(u)$. The corresponding eigenvalue is determined by (3.50), (7.3) and $\mathbf{G}'(0) = 1$. The eigenvalue of the noncommutative Macdonald polynomial \mathbf{Q}'_λ follows from the definition (3.52) employing the automorphism ω_t in (2.29). ■

Having derived the formal conditions (7.2) on the Bethe roots for the Bethe vector (7.1) to be an eigenvector, we need to investigate if solutions to the set of equations (7.2) do exist and if any, how many and their properties.

7.2 Completeness of the Bethe ansatz

In this section we are going to prove for t an indeterminate that the Bethe vectors (7.1) form eigenbasis; see Theorem 7.8 below. The strategy of the proof is entirely different from the one used in [73] where t is assumed to be a real parameter in the interval $[-1, 1]$. Here we will introduce an algebraic variety and its coordinate ring showing that the associated Hilbert polynomial is a constant, the dimension of the k particle subspace $\mathcal{F}_k^{\otimes n}$. In other words, the solutions to (7.2) are a 0-dimensional variety, a set of discrete points, with cardinality $|\mathcal{A}_{k,n}^+| = \dim \mathcal{F}_k^{\otimes n}$. In the next section we will then identify this coordinate ring as a deformation of the Verlinde ring. We start with some general observations concerning the possible solutions to (7.2).

Permutation invariance. First we note that the Bethe vectors (7.1) are symmetric in the Bethe roots, $\mathbf{b}(y) = \mathbf{b}(wy)$ for any $w \in \mathfrak{S}_k$ because of (3.28). We have already exploited this earlier; see Corollary 4.5. Thus, we can identify each solution $y = (y_1, \dots, y_k)$ to the Bethe ansatz equations (7.2) with the set $\{y_1, \dots, y_k\}$ or its orbit under the natural action of the symmetric group.

Scale invariance. Henceforth we will assume $z^{\pm 1/2n}$ to exist. The z -dependence of the Bethe equations (7.2) can be eliminated by rescaling the Bethe roots as $y \rightarrow z^{-1/n}y$, hence, without loss of generality we will repeatedly set $z = 1$ as it simplifies the computations. By the same token we note that there is a \mathbb{Z}_n -action which leaves the set of solutions invariant: for any $\ell \in \mathbb{Z}_n$ and any solution $y = (y_1, \dots, y_k)$ the rescaled solution $e^{2\pi i \ell/n}y = (e^{2\pi i \ell/n}y_1, \dots, e^{2\pi i \ell/n}y_k)$ also satisfies the Bethe ansatz equations.

Inversion and complex conjugation. Finally, if $y = (y_1, \dots, y_k)$ is a solution, then one easily checks that $y^{-1} = (y_1^{-1}, \dots, y_k^{-1})$ is a solution for $z \rightarrow z^{-1}$. If we make the following further assumptions on the indeterminates t and z ,

$$\bar{t} = t \quad \text{and} \quad \bar{z}^{\pm \frac{1}{2n}} = z^{\mp \frac{1}{2n}} \quad (7.6)$$

then we find that also $\bar{y} = (\bar{y}_1, \dots, \bar{y}_k)$ is a solution with $z \rightarrow z^{-1}$. This latter assumption is physically motivated: it makes the quantum Hamiltonians (5.4) Hermitian; see (7.9) below.

We will now exploit the permutation invariance of the Bethe ansatz equations (7.2) by reformulating them in terms of symmetric functions in the Bethe roots. This will lead us to the definition of an algebraic variety

and its vanishing ideal in the ring of symmetric functions. Recall from the definition (2.16) that

$$g_r(y; 0, t) = Q_{(r)}(y; 0, t) = (1 - t) \sum_{i=1}^k y_i^r \prod_{j \neq i} \frac{y_i - y_j t}{y_i - y_j}$$

is a symmetric polynomial of degree r . For ease of notation, we will henceforth denote $g_r(y; 0, t)$ simply by $g_r(y; t)$.

Lemma 7.3 (i) *The Bethe ansatz equations (7.2) are equivalent to the equations*

$$g_n(y; t) = z(1 - t^k) \quad \text{and} \quad g_{n+r}(y; t) + zt^k g_r(y; t^{-1}) = 0, \quad 0 < r < k, \quad (7.7)$$

(ii) *Let y be a solution to the Bethe ansatz equations (7.2). Then one has the identities*

$$g_{n-r}(y_1, \dots, y_k; t) = z g_r(y_1^{-1}, \dots, y_k^{-1}; t), \quad 0 < r < n. \quad (7.8)$$

Proof. To prove the first assertion (i) assume that $y = (y_1, \dots, y_k)$ satisfy the Bethe ansatz equations (7.2). Recall $\prod_{i=1}^k \frac{1-u}{1-uy_i} = \sum_{r \geq 0} g_r(y; t) u^r$ with $g_0(y; t) = 1$. Using the coproduct of the Yang-Baxter algebra one shows easily by induction that $e_n = z \cdot 1$ and that $e_r = 0$ for $r > n$. The assertion now follows from (7.3).

Assume now that $y = (y_1, \dots, y_k)$ are invertible and such that (7.7) hold. Let $E(u) = \prod_{i=1}^k (1 + u y_i)$ be the generating function of the elementary symmetric functions $e_r(y)$ and $G(u) = E(-ut)/E(-u)$ be the generating function of the $g_r(y; t)$'s. Then $E(-u)G(u) = E(-ut)$ and, hence,

$$\sum_{a+b=r} (-1)^a e_a(y) g_b(y; t) = (-t)^r e_r(y).$$

Employing this relation one calculates (recall that $n > 2$ and that $e_r(y) = 0$ for $r > k$)

$$\begin{aligned} 0 &= e_1(g_{n+k-1}(t) + zt^k g_{k-1}(t^{-1})) = g_{n+k}(t) + zt^k g_k(t^{-1}) + z(-1)^{k-1} e_k \\ &\quad + e_2(g_{n+k-2}(t) + zt^k g_{k-2}(t^{-1})) - e_3(g_{n+k-3}(t) + zt^k g_{k-3}(t^{-1})) + \dots \\ &\quad \dots + (-1)^k e_k(g_n(t) + zt^k) = g_{n+k}(t) + zt^k g_k(t^{-1}), \end{aligned}$$

where we have used in the last line (7.7). Similarly one shows by induction that (7.7) imply $g_{n+r}(t) + zt^k g_r(t^{-1}) = 0$ for any $r \geq k$. Set $f = f(u)$ to be the eigenvalue appearing in (7.3). It now follows that f is polynomial in u , since $g_{n+r}(t) + zt^k g_r(t^{-1}) = 0$ for all $r \geq k$ ensures that the formal power series expansion of f with respect to u terminates after finitely many terms. Thus, the residues of f at $u = y_i^{-1}$ for $i = 1, \dots, k$ have to vanish. These conditions are equivalent to (7.2).

The second assertion (ii) follows from a simple computation,

$$\begin{aligned} g_{n-r}(y; t) &= (1 - t) \sum_{i=1}^k y_i^{n-r} \prod_{j \neq i} \frac{y_i - y_j t}{y_i - y_j} = z(1 - t) \sum_{i=1}^k y_i^{-r} \prod_{j \neq i} \frac{y_i t - y_j}{y_i - y_j} \\ &= z(1 - t) \sum_{i=1}^k y_i^{-r} \prod_{j \neq i} \frac{y_i^{-1} - y_j^{-1} t}{y_i^{-1} - y_j^{-1}} = z g_r(y_1^{-1}, \dots, y_k^{-1}; t). \end{aligned}$$

■

Remark 7.1 Using the relation (ii) the spectrum of the commuting quantum Hamiltonians (5.4) of the q -boson model as

$$H_r^\pm \mathbf{b}(y) = -\frac{g_r(y_1, \dots, y_k; t) \pm z g_r(y_1^{-1}, \dots, y_k^{-1}; t)}{2} \mathbf{b}(y). \quad (7.9)$$

Assuming (7.6) implies that $z^{-1/2} H_r^\pm$ are (anti-) Hermitian operators.

Recall that the functions $g_r(t) = g_r(0, t)$ are elements in $\mathbb{Z}[t][e_1, \dots, e_k]$ where the e_r are the elementary symmetric functions in k variables; see (2.15). Thus, we can interpret (7.7) as a set of polynomial equations in the variables $\{e_1, \dots, e_k\}$ with coefficients in $\mathbb{Z}[t]$.

Example 7.1 Employing that

$$g_r(t) = \sum_{s=0}^r h_{r-s} e_s (-t)^s, \quad h_r = \det(e_{1-i+j})$$

the equations (7.7) can be reformulated in terms of elementary symmetric polynomials leading to a coupled set of algebraic equations. For instance, set $n = 3$ and $k = 2$. From the Bethe ansatz equations (7.2) one easily derives that $e_k(y)^n = 1$; this latter relation is true for general n, k . Thus, in the present example with $k = 2$ we only need an additional relation to express e_1 in terms of $e_{k=2}$. Using the expansion of the equation $z g_1(y^{-1}; t) = g_2(y; t)$ into elementary symmetric polynomials one arrives for $n = 3$, $k = 2$ and $z = 1$ at the quadratic equation

$$e_1^2 - e_1 e_2^2 - (1+t) e_2 = 0$$

Here we have used that $e_r(y^{-1}) = e_k^{n-1}(y) e_{k-r}(y)$.

As this example shows we are led to solving polynomial equations with coefficients in $\mathbb{Z}[t]$. In order to guarantee the existence of solutions we must work over an algebraically closed field. This motivates us to consider the algebraically closed field of Puiseux series,

$$\mathbb{k} = \mathbb{C}\{\{t\}\} := \bigcup_{m=1}^{\infty} \mathbb{C}((t^{1/m})), \quad (7.10)$$

which is the formal union of the fields of Laurent series in $t^{1/m}$. That is, an element $f \in \mathbb{k}$ is a formal expression of the form $f = \sum_{\ell \geq \ell_0} c_\ell t^{\ell/m}$ with $c_\ell \in \mathbb{C}$.

Fix $k > 0$ and set $\tilde{g}_{n+r}(t) = g_{n+r}(t) + z t^k g_r(t^{-1})$ for $r = 1, \dots, k-1$, $\tilde{g}_n(t) = g_n(t) - z(1-t^k)$ and $\tilde{g}_r(t) = g_r(t)$ for $r \notin \{n, n+1, \dots, n+k-1\}$. In what follows we suppress for simplicity the explicit dependence on t in the notation and set once more $z = 1$. Denote by

$$\mathbf{V}_{k,n} := \{\epsilon = (\epsilon_1, \dots, \epsilon_k) \in \mathbb{k}^k : \tilde{g}_n|_{\epsilon=\epsilon} = \dots = \tilde{g}_{n+k-1}|_{\epsilon=\epsilon} = 0\} \quad (7.11)$$

the solutions to (7.7) for $z = 1$ in the affine space \mathbb{k}^k , where $\tilde{g}_r|_{\epsilon=\epsilon}$ is obtained by replacing $e_r \rightarrow \epsilon_r$ in its polynomial expression $\tilde{g}_n = \sum_\lambda c_\lambda e_{\lambda_1} \dots e_{\lambda_\ell}$.

Lemma 7.4 Let $\mathbf{I}(\mathbf{V}_{k,n}) \subset \mathbb{k}[e_1, \dots, e_k]$ be the vanishing ideal of the affine variety (7.11) and define the two-sided ideal $\mathbf{I}_{k,n} := \langle \tilde{g}_n, \dots, \tilde{g}_{n+k-1} \rangle$. Then

$$\mathbf{I}(\mathbf{V}_{k,n}) = \mathbf{I}_{k,n}. \quad (7.12)$$

In other words, given two symmetric functions $f, g \in \mathbb{k}[e_1, \dots, e_k]$ their difference $f - g$ lies in the ideal $\mathbf{I}_{k,n}$ if and only if $f(y) = g(y)$ for all solutions y of the Bethe ansatz equations (7.2). The proof of this statement follows a similar strategy as the one given in [40, Proof of Theorem 6.20, Claim 1] for the Verlinde algebra.

Proof. We show that the ideal $\mathbf{I}_{k,n}$ is radical, i.e. $\mathbf{I}_{k,n} = \sqrt{\mathbf{I}_{k,n}}$, and the assertion then follows from the strong Nullstellensatz. The results in [53, (2.16), p. 213] imply that $\Lambda_k(t) := \mathbb{k}[e_1, \dots, e_k] \cong \mathbb{k}[g_1, \dots, g_k]$ and in the projective limit $\Lambda(t) = \varprojlim \Lambda_k(t)$ we have $\mathbb{k}[g_1, g_2, \dots] \cong \mathbb{k}[e_1, e_2, \dots]$. In particular the g_r 's are all algebraically independent and, thus, the \tilde{g}_r also form an algebraically independent set (note that g_{n+r} and g_r have different degree). Hence, the elements in $\{\tilde{g}_\lambda := \tilde{g}_{\lambda_1} \tilde{g}_{\lambda_2} \cdots\}_\lambda$, where λ ranges over all partitions, are linearly independent. Suppose $f = \sum_\lambda c_\lambda \tilde{g}_\lambda$ with $f \in \mathbb{k}[e_1, \dots, e_k] \subset \mathbb{k}[e_1, e_2, \dots]$ is not in $\mathbf{I}_{k,n}$, then there must exist at least one partition μ such that $\mu_j \notin \{n, n+1, \dots, n+k-1\}$ for all j and $c_\mu \neq 0$. We can therefore conclude that the expansion of f^m , $m \geq 1$ contains \tilde{g}_{μ^m} where μ^m is the partition containing each part $\mu_j > 0$ exactly m times. Thus, $f^m \notin \mathbf{I}_{k,n}$ and projecting onto $\mathbb{k}[e_1, \dots, e_k]$ now yields the desired result. ■

It will be important to compute which Hall-Littlewood functions lie in the ideal $\mathbf{I}_{k,n}$. Recall the definition of the raising ($i < j$) and lowering ($i > j$) operators $R_{ij}\lambda := (\dots, \lambda_i + 1, \dots, \lambda_j - 1, \dots)$ and $R_{ij}f_\lambda := f_{R_{ij}\lambda}$. Then, in principle, one can calculate whether a Hall-Littlewood Q -function lies in $\mathbf{I}_{k,n}$ by employing the identity (2.23) [53]. However, a simpler approach is given by making use of the alternative definition of Hall-Littlewood functions via the symmetric group. Namely, introduce another function $R_\lambda = R_\lambda(x_1, \dots, x_k; t)$ via the equalities

$$(1-t)^k R_\lambda = P_\lambda \prod_{i=0}^{\lambda_1} (t)_{m_i(\lambda)} \quad \text{and} \quad (1-t)^k R_\lambda = (t)_{k-\ell(\lambda)} Q_\lambda. \quad (7.13)$$

The function R_λ for λ a *partition* can be generalised to *compositions* μ via [53]

$$R_\mu(x_1, \dots, x_k; t) := \sum_{w \in \mathfrak{S}_k} w \left(x^\mu \prod_{i < j} \frac{x_i - tx_j}{x_i - x_j} \right) \quad (7.14)$$

The latter are a linear combination of the functions R_λ indexed by *partitions* λ which is obtained by repeatedly applying the following “straightening rules” [53]

$$R_{\lambda, \sigma_i} = tR_\lambda - R_{(\dots, \lambda_i - 1, \lambda_{i+1} + 1, \dots)} + tR_{(\dots, \lambda_{i+1} + 1, \lambda_i - 1, \dots)}, \quad i = 1, \dots, k-1 \quad (7.15)$$

and

$$R_{(\lambda_1, \dots, \lambda_k)} = 0, \quad \lambda_k < 0. \quad (7.16)$$

The following lemma now shows that the Bethe ansatz equations (7.2) extend these straightening rules to the extended affine symmetric group $\hat{\mathfrak{S}}_k$, that is any R_μ with $\mu \in \mathcal{P}_k$ can be written as a linear combination of R_λ with $\lambda \in \mathcal{A}_{k,n}^+$. We therefore can say that this particular model exhibits an extended affine Weyl group invariance or ‘symmetry’.

Lemma 7.5 (extended affine straightening rules) *Let $\lambda \in \mathcal{A}_{k,n}^+$. Then the following polynomials are in the ideal $\mathbf{I}_{k,n}$*

$$R_{\lambda, \sigma_0} - tR_\lambda + R_{(\lambda_1 + 1, \dots, \lambda_k - 1)} - tR_{(\lambda_k - 1 + n, \dots, \lambda_1 + 1 - n)} \quad (7.17)$$

and

$$R_\lambda - zR_{\lambda,\tau}, \quad \lambda_1 \geq n, \quad (7.18)$$

where σ_0, τ are the additional generators of the extended affine symmetric group $\hat{\mathfrak{S}}_k$ whose right action on λ is given by (2.5).

Proof. Let $\theta(t) := \prod_{1 \leq i < j \leq k} \frac{y_i - ty_j}{y_i - y_j}$ and $\sigma_{ij} := \sigma_{j-1}\sigma_{j-2} \cdots \sigma_{i+1}\sigma_i$. From the Bethe ansatz equations

$$1 = zy_1^{-n} \prod_{j \neq 1} \frac{y_1 t - y_j}{y_1 - ty_j} = z^{-1} y_k^n \prod_{j \neq 1} \frac{y_j t - y_k}{y_j - ty_k}$$

one derives the identity

$$(y_1 - ty_k) y^{\lambda \cdot \sigma_0} \theta(t) = -\sigma_{1k} [(y_1 - ty_k) y^\lambda \theta(t)]$$

and the first affine straightening rule now follows from Lemma 7.4.

To prove the second rule we first note the relations

$$\begin{aligned} \sigma_i \theta(t) &= \frac{y_i t - y_{i+1}}{y_i - ty_{i+1}} \theta(t), \\ \sigma_{ij} \theta(t) &= \frac{y_i t - y_j}{y_i - ty_j} \frac{y_{i+1} t - y_j}{y_{i+1} - ty_j} \cdots \frac{y_{j-1} t - y_j}{y_{j-1} - ty_j} \theta(t), \\ \sigma_{ij}^{-1} \theta(t) &= \frac{y_i t - y_j}{y_i - ty_j} \frac{y_i t - y_{j-1}}{y_i - ty_{j-1}} \cdots \frac{y_i t - y_{i+1}}{y_i - ty_{i+1}} \theta(t). \end{aligned}$$

Hence, it follows from the Bethe ansatz equations that

$$\sigma_{ik}^{-1} \theta(t) = \left(\prod_{i < j} \frac{ty_i - y_j}{y_i - ty_j} \right) \theta(t) \stackrel{\text{BAE}}{=} z^{-1} y_i^n \left(\prod_{j < i} \frac{y_i - ty_j}{ty_i - y_j} \right) \theta(t) = z^{-1} \sigma_{1i} [y_1^n \theta(t)].$$

In particular, choosing $i = 1$ we obtain the identity

$$\begin{aligned} y_1^n \theta(t) &= z \sigma_{1k}^{-1} \theta(t) = z \sigma_1 \sigma_2 \cdots \sigma_{k-1} \theta(t), \\ \Rightarrow y^\lambda \theta(t) &= z \sigma_1 \sigma_2 \cdots \sigma_{k-1} \left(y_1^{\lambda_2} \cdots y_{k-1}^{\lambda_k} y_k^{\lambda_1 - n} \theta(t) \right) \end{aligned}$$

which proves the second straightening rule applying once more Lemma 7.4. ■

Remark 7.2 Note in particular that for $\lambda = (\lambda_1, \dots, \lambda_k)$ with $\lambda_1 = n$ and $\lambda_k > 0$ the polynomials

$$P_\lambda - z^{m_n(\lambda)} P_{\tilde{\lambda}} \quad \text{and} \quad Q_\lambda - z^{m_n(\lambda)} (t)_{m_n(\lambda)} Q_{\tilde{\lambda}}, \quad (7.19)$$

where $\tilde{\lambda}$ is the reduced partition with all parts n removed lie in the ideal $\mathbf{I}_{k,n}$.

The relation between the extended affine straightening rules and the definition of $\mathbf{I}_{k,n}$ is given by the following result.

Lemma 7.6 We have for $r > 0$ that

$$\frac{(1-t)^k}{(t)_{k-1}} R_{(0, \dots, 0, r)} = -t^k g_r(t^{-1}). \quad (7.20)$$

Proof. Inserting the definition of the polynomial R_μ we obtain

$$\begin{aligned} R_{(0,\dots,0,r)} &= \sum_{w \in \mathfrak{S}_k} w \left(x_k^r \prod_{i < j} \frac{x_i - tx_j}{x_i - x_j} \right) \\ &= \sum_{i=1}^k x_i^r \prod_{j \neq i} \frac{x_i t - x_j}{x_i - x_j} \sum_{w \in \mathfrak{S}_{k-1}} w \left(\prod_{i < j < k} \frac{x_i - tx_j}{x_i - x_j} \right) \\ &= \frac{(t)_{k-1}}{(1-t)^{k-1}} \sum_{i=1}^k x_i^r \prod_{j \neq i} \frac{x_i t - x_j}{x_i - x_j} = -\frac{(t)_{k-1} t^k}{(1-t)^k} g_r(t^{-1}), \end{aligned}$$

where in the second equality in the first line the sum only runs over permutations $w \in \mathfrak{S}_{k-1}$ which only permute the first $k-1$ indices. ■

Proposition 7.7 (Basis of the coordinate ring) *The quotient $\mathbb{k}[\mathbf{V}_{k,n}] := \mathbb{k}[e_1, \dots, e_k] / \mathbf{I}_{k,n}$ viewed as a vector space has dimension $|\mathcal{A}_{k,n}^+| = \binom{n-1}{k}$. A basis is given by the equivalence classes of the Hall-Littlewood functions $P_{\tilde{\lambda}}$ with $\tilde{\lambda} \in \tilde{\mathcal{A}}_{k,n}^+$.*

Proof. The proof is a generalisation of the one in [40, Theorem 6.20, Claim 4]. Recall that the Hall-Littlewood functions $\{P_\lambda : \lambda \text{ partition with } \ell(\lambda) \leq k\}$ form a $\mathbb{Z}[t]$ -basis of $\mathbb{Z}[t][e_1, \dots, e_k]$; this follows from [53, (2.7), p. 209] and projecting onto the ring of symmetric function with k -variables by setting $e_r = 0$ for $r > k$. Denote by $[P_\lambda] := P_\lambda + \mathbf{I}_{k,n}$ the equivalence class of P_λ in $\mathbb{k}[\mathbf{V}_{k,n}]$. Then it follows from our previous lemma that for any partition $\lambda \in \mathcal{P}_k^+$ there exists a $\mu \in \mathfrak{S}_k \tilde{\mathcal{A}}_{k,n}^+$ such that $[P_\lambda] = [P_\mu]$. Using the (non-affine) straightening rules (7.15) and (7.16) P_μ can be written as $\mathbb{Z}[t]$ -linear combination of $P_{\tilde{\nu}}$'s with $\tilde{\nu} \in \tilde{\mathcal{A}}_{k,n}^+$. Thus, we have that the vector space dimension of $\mathbb{k}[\mathbf{V}_{k,n}]$ must be smaller or equal than $|\tilde{\mathcal{A}}_{k,n}^+| = |\mathcal{A}_{k,n}^+|$.

Now assume that $0 = \sum_{\tilde{\lambda} \in \tilde{\mathcal{A}}_{k,n}^+} c_{\tilde{\lambda}} [P_{\tilde{\lambda}}]$ which is equivalent to $\sum_{\tilde{\lambda} \in \tilde{\mathcal{A}}_{k,n}^+} c_{\tilde{\lambda}} P_{\tilde{\lambda}} = \sum_{r=n}^{n+k-1} f_r \tilde{g}_r$ for some $f_r \in \mathbb{k}[e_1, \dots, e_k]$. The transition matrix between the basis $\{P_\lambda\}$ and the basis $\{g_\lambda\}$ in the ring of symmetric functions is strictly lower unitriangular with respect to the natural or dominance partial ordering [53, (2.16), p. 213], whence we can conclude that the last equality can be written as $\sum_{\tilde{\lambda} \in \tilde{\mathcal{A}}_{k,n}^+} c_{\tilde{\lambda}} g_{\tilde{\lambda}} = \sum_{\mu \notin \tilde{\mathcal{A}}_{k,n}^+} d_\mu \tilde{g}_\mu$ for some $d_\mu \in \mathbb{k}$. Recall that the \tilde{g}_λ 's are also linearly independent and that $\tilde{g}_{\tilde{\lambda}} = g_{\tilde{\lambda}}$ for $\tilde{\lambda} \in \tilde{\mathcal{A}}_{k,n}^+$. Thus, $c_{\tilde{\lambda}} = 0$ for all $\tilde{\lambda} \in \tilde{\mathcal{A}}_{k,n}^+$ which establishes linear independence of the $P_{\tilde{\lambda}}$ with $\tilde{\lambda} \in \tilde{\mathcal{A}}_{k,n}^+$ in $\mathbb{k}[\mathbf{V}_{k,n}]$. ■

We are now ready to prove the main statement of this section.

Theorem 7.8 (completeness of the Bethe ansatz) *Let t be an indeterminate and set $z = 1$. (1) There is a bijection $\sigma \mapsto y_\sigma = (y_1, \dots, y_k) \in \mathbb{k}^k$ between partitions in $\mathcal{A}_{k,n}^+$ and solutions to the Bethe ansatz equations (7.2). We denote the Bethe vector (7.1) corresponding to $\sigma \in \mathcal{A}_{k,n}^+$ by $\mathbf{b}_\sigma := \mathbf{b}(y_\sigma)$. (2) The Bethe states $\{\mathbf{b}_\sigma \mid \sigma \in \mathcal{A}_{k,n}^+\}$ provide an orthogonal basis of $\mathcal{F}_k^{\otimes n} \otimes_{\mathbb{C}(t)} \mathbb{k}$, i.e. one has the identity*

$$\langle \mathbf{b}_\rho | \mathbf{b}_\sigma \rangle = \sum_{\lambda \in \mathcal{A}_{k,n}^+} Q_\lambda(y_\rho; t) P_\lambda(\bar{y}_\sigma; t) = 0,$$

where $\rho \neq \sigma$ are two distinct partitions in $\mathcal{A}_{k,n}^+$.

Proof. Without loss of generality we can set again $z = 1$. It follows from our previous lemma, $I(\mathbf{V}_{k,n}) = I_{k,n}$, that the dimension of the algebraic variety $\mathbf{V}_{k,n}$ equals the degree of the (affine) Hilbert polynomial of $I_{k,n}$; see e.g. [15, Definitions 5,7 and Theorem 8, Chapter 9, §3, pp. 459-461]. But since $\mathbb{k}[e_1, \dots, e_k]/I_{k,n} = \mathbb{k}[\mathbf{V}_{k,n}]$ has finite dimension as a vector space, the affine Hilbert polynomial of $I_{k,n}$ is of degree zero, i.e. a constant and this constant equals the vector space dimension [15, Chapter 9, §4, Ex 10, p.475] which we showed to be $|\mathcal{A}_{k,n}^+|$. Furthermore, $\mathbf{V}_{k,n}$ is non-empty. To see this assume first that $\mathbf{V}_{k,n} = \emptyset$. Then the Hilbert polynomial of $I(\mathbf{V}_{k,n})$ would be the zero polynomial [15, p. 461], but we have just seen that it is a nonzero constant. Hence, we can conclude that $\mathbf{V}_{k,n}$ consists of finitely many points and, furthermore, $|\mathbf{V}_{k,n}|$ equals the (constant) Hilbert polynomial of $\mathbb{k}[\mathbf{V}_{k,n}]$; see for e.g. [15, Chapter 9, §4, Prop 6, p. 471 and Ex 11, p. 475]. Hence, we arrive at $|\mathbf{V}_{k,n}| = |\mathcal{A}_{k,n}^+|$ meaning that there are as many distinct solutions to the Bethe ansatz equations as the dimension of the subspace $\subset \mathcal{F}^{\otimes n}$ spanned by $\{|\lambda\rangle : \lambda \in \mathcal{A}_{k,n}^+\}$.

The second claim, that the Bethe vectors are orthogonal, now follows from observing that the eigenvalues of the transfer matrices (3.42), (3.46) separate points, which means that for $\rho, \sigma \in \mathcal{A}_{k,n}^+$ with $\rho \neq \sigma$ the corresponding eigenvalues have to be different. From this fact one now easily deduces that the corresponding scalar product between the eigenvectors has to vanish. ■

We conclude this section by stating two more technical results which are related to the behaviour of the Bethe roots under taking the inverse and complex conjugation. They will be needed when introducing a Frobenius structure on the coordinate ring $\mathbb{k}[\mathbf{V}_{k,n}]$ in the next section.

Lemma 7.9 (Inversion property) *Let $\lambda \in \mathcal{A}_{k,n}^+$ and $y = (y_1, \dots, y_k)$ be a solution to the Bethe ansatz equations. Then*

$$R_\lambda(y; t) = z^k R_{(n-\lambda_k, \dots, n-\lambda_2, n-\lambda_1)}(y^{-1}; t) \quad (7.21)$$

and, thus, setting $z = 1$ we have $P_\lambda(y; t) = P_{\lambda^}(y^{-1}; t)$ as well as $Q_\lambda(y; t) = Q_{\lambda^*}(y^{-1}; t)$ with λ^* being the inverse image of $(n - \lambda_k, \dots, n - \lambda_2, n - \lambda_1) \in \tilde{\mathcal{A}}_{k,n}^+$ under the bijection $\sim : \mathcal{A}_{k,n}^+ \rightarrow \tilde{\mathcal{A}}_{k,n}^+$.*

Proof. From the Bethe ansatz equations we have that $y_1^n \cdots y_k^n = z^k$. Furthermore, one easily verifies that

$$w_k \left(\prod_{1 \leq i < j \leq k} \frac{y_i - ty_j}{y_i - y_j} \right) = \prod_{1 \leq i < j \leq k} \frac{y_i t - y_j}{y_i - y_j} = \prod_{1 \leq i < j \leq k} \frac{y_i^{-1} - ty_j^{-1}}{y_i^{-1} - y_j^{-1}}.$$

Hence, we arrive at the identity

$$y^\lambda \prod_{1 \leq i < j \leq k} \frac{y_i - ty_j}{y_i - y_j} = w_k \left(y_1^{\lambda_k - n} \cdots y_{k-1}^{\lambda_2 - n} y_k^{\lambda_1 - n} \prod_{1 \leq i < j \leq k} \frac{y_i t - y_j}{y_i - y_j} \right)$$

which gives the desired equation for R_λ . Exploiting the definition (7.13) of P_λ in terms of R_λ , the remaining identities follow from $P_\lambda(y; t) = z^{m_n(\lambda)} P_{\tilde{\lambda}}(y; t)$ and the obvious fact that $b_\lambda(t) = b_{\lambda^*}(t)$. ■

Lemma 7.10 (dual Bethe vectors) *Assume (7.6) holds. For any solution $y = (y_1, \dots, y_k)$ of the Bethe ansatz equations we have that $P_{\tilde{\lambda}}(\bar{y}^{-1}; t) = P_{\tilde{\lambda}}(y; t)$. This in particular, implies that the dual Bethe vectors*

are given by

$$\mathbf{b}_\sigma^* = \frac{1}{\|\mathbf{b}_\sigma\|^2} \sum_{\lambda \in \mathcal{A}_{k,n}^+} P_{\bar{\lambda}}(y; t) \langle \lambda |, \quad \|\mathbf{b}_\sigma\|^2 = \sum_{\lambda \in \mathcal{A}_{k,n}^+} Q_\lambda(y_\sigma; t) P_\lambda(\bar{y}_\sigma; t), \quad (7.22)$$

that is, we have the identity $\langle \mathbf{b}_\rho^* | \mathbf{b}_\sigma \rangle = \delta_{\rho\sigma}$.

Proof. Assume (7.6) holds. Then we can assume without loss of generality that $z = 1$. It then follows from (5.6) that the quantum Hamiltonians H_r^\pm defined in (5.4) are (anti-)Hermitian and, hence, we infer from (7.9) that $g_r(y; t) = g_r(\bar{y}^{-1}; t)$ for $r = 1, \dots, n-1$. Observing that $t^k g_r(y; t^{-1})$ is a polynomial of the $g_r(y; t)$'s for $r = 1, \dots, n-1$ (this follows from (7.20)) we can employ (7.7) to conclude that $g_r(y; t) = g_r(\bar{y}^{-1}; t)$ for $r \geq n$. Hence, we find $Q_{\bar{\lambda}}(y; t) = Q_{\bar{\lambda}}(\bar{y}^{-1}; t)$ and $P_{\bar{\lambda}}(\bar{y}^{-1}; t) = P_{\bar{\lambda}}(y; t)$ since the latter are polynomials in the g_r 's. Noting that $Q_{\bar{\lambda}}, P_{\bar{\lambda}}$ are homogeneous functions of degree $|\bar{\lambda}|$ the z -dependence is easily re-introduced. ■

7.3 Deformed fusion coefficients and Frobenius structures

This section will see the formulation of the main result of this article: there exists a natural Frobenius algebra structure on the Kirillov-Reshetikhin module $W^{1,k}$ which we have previously identified with the k -particle Fock space $\mathcal{F}_k^{\otimes n}$; see Proposition 3.5. We are going to extend the base field from $\mathbb{C}(t)$ to \mathbb{k} as this will allow us to include the idempotents.

Theorem 7.11 (deformed Verlinde algebra) *Let $\mathfrak{F}_{n,k} := \mathcal{F}_k^{\otimes n} \otimes_{\mathbb{C}(t)} \mathbb{k}$ and set $z = 1$. Define for $\lambda, \mu \in \mathcal{A}_{k,n}^+$ the product*

$$|\lambda\rangle \circledast |\mu\rangle := \mathbf{Q}'_{\lambda'} |\mu\rangle \quad (7.23)$$

and the bilinear form $\eta : \mathfrak{F}_{n,k} \otimes \mathfrak{F}_{n,k} \rightarrow \mathbb{k}$

$$\eta(|\lambda\rangle \otimes |\mu\rangle) := \delta_{\lambda\mu^*} / b_\lambda(t). \quad (7.24)$$

Then $(\mathfrak{F}_{n,k}, \circledast, \eta)$ is a commutative Frobenius algebra with unit $|n^k\rangle$.

Remark 7.3 *Implicit in the last theorem is the statement that $\mathbf{Q}'_{\lambda'} |n^k\rangle = |\lambda\rangle$. Moreover, one checks from the definition (3.46), (3.47) that $\mathbf{g}'_r |n^k\rangle = 0$ for $r > k$ and, hence, $\mathbf{Q}'_{\lambda'} |n^k\rangle = 0$ for $\lambda \notin \mathcal{A}_{k,n}^+$. That is, the family $\mathbf{B}_n := \{\mathbf{Q}'_\lambda : \lambda \in \mathcal{P}_n^+\} \subset \mathcal{H}_q^{\otimes n}$ of noncommutative analogues of Macdonald polynomials generates the canonical basis in the U_n -module $S^k(V) \subset V^{\otimes k}$ when acting on the highest weight vector $|n^k\rangle = v_n \otimes \dots \otimes v_n$.*

The proof of these statements will employ the expression of the matrix elements $\langle \lambda | \mathbf{Q}'_{\lambda'} | \mu \rangle$ in terms of the Bethe vectors (7.1) which we state as a separate lemma. To ease the notation we introduce the transition matrix

$$\mathcal{S}_{\lambda\mu}(t) := \|\mathbf{b}_\mu\| \langle \mathbf{b}_\mu^* | \lambda \rangle = \frac{P_\lambda(y_\mu; t)}{\|\mathbf{b}_\mu\|} \quad (7.25)$$

from the basis of *normalised* Bethe vectors to the vectors $\{|\lambda\rangle : \lambda \in \mathcal{A}_{k,n}^+\}$ which we identified with the canonical basis in $S^k(V)$. The matrix elements of the inverse matrix $\mathcal{S}^{-1}(t)$ are given by

$$\mathcal{S}_{\mu\lambda}^{-1}(t) = \frac{\langle \lambda | \mathbf{b}_\mu \rangle}{\|\mathbf{b}_\mu\|} = \frac{Q_\lambda(y_\mu^{-1}; t)}{\|\mathbf{b}_\mu\|} = b_\lambda(t) \overline{\mathcal{S}_{\lambda\mu}(t)} = b_\lambda(t) z^{\frac{|\lambda| - |\lambda^*|}{n}} \mathcal{S}_{\lambda^* \mu}(t), \quad (7.26)$$

where we have used (7.21), (7.22) in the last two equalities under the assumption (7.6). Labelling the partition n^k with ‘0’ note that we have in particular $\mathcal{S}_{0\mu}(t) = \|\mathbf{b}_\mu\|^{-1}$ because of (7.19).

Lemma 7.12 (deformed Verlinde formulae) *Let $\lambda, \mu, \nu \in \mathcal{A}_{k,n}^+$. Then we have*

$$\langle \lambda | \mathbf{Q}'_{\nu'} | \mu \rangle = z^{\frac{|\mu|+|\nu|-|\lambda|}{n}} \sum_{\sigma \in \mathcal{A}_{k,n}^+} \frac{\mathcal{S}_{\mu\sigma}(t) \mathcal{S}_{\nu\sigma}(t) \mathcal{S}_{\sigma\lambda}^{-1}(t)}{\mathcal{S}_{0\sigma}(t)} \quad (7.27)$$

and

$$\langle \lambda | \mathbf{S}'_{\nu'} | \mu \rangle = z^{\frac{|\mu|+|\nu|-|\lambda|}{n}} \sum_{\sigma \in \mathcal{A}_{k,n}^+} s_{\nu'}(y_\sigma) \mathcal{S}_{\nu\sigma}(t) \mathcal{S}_{\sigma\lambda}^{-1}(t). \quad (7.28)$$

Both coefficients vanish identically unless $d = \frac{|\mu|+|\nu|-|\lambda|}{n} \in \mathbb{Z}_{\geq 0}$.

Proof. Denote by $y_\sigma = z^{1/n} y'_\sigma$ the solution to (7.2) under the bijection of Theorem 7.8. Then we have the following simple calculation,

$$\begin{aligned} \langle \lambda | \mathbf{Q}'_{\nu'} | \mu \rangle &= \sum_{\sigma \in \mathcal{A}_{k,n}^+} \langle \lambda | \mathbf{Q}'_{\nu'} | \mathbf{b}_\sigma \rangle \langle \mathbf{b}_\sigma^* | \mu \rangle = \sum_{\sigma \in \mathcal{A}_{k,n}^+} z^{\frac{|\nu|}{n}} P_{\nu'}(y'_\sigma; t) \langle \lambda | \mathbf{b}_\sigma \rangle \langle \mathbf{b}_\sigma^* | \mu \rangle \\ &= z^{\frac{|\mu|+|\nu|-|\lambda|}{n}} \sum_{\sigma \in \mathcal{A}_{k,n}^+} \frac{P_{\bar{\mu}}(y'_\sigma; t) P_{\nu'}(y'_\sigma; t) Q_{\lambda^*}(y'_\sigma; t)}{\|\mathbf{b}_\sigma\|^2} \end{aligned} \quad (7.29)$$

and the first assertion now follows from the definition (7.25). Here we have used in the first line that the Bethe vectors form an eigenbasis in $\mathcal{F}_k^{\otimes n}$ and in the second line the explicit expansions (7.1), (7.22) of them and their dual vectors with respect to the basis $\{|\lambda\rangle\}_{\lambda \in \mathcal{A}_{k,n}^+}$ as well as (7.19). The second identity for $\langle \lambda | \mathbf{S}'_{\nu'} | \mu \rangle$ follows from a computation along the same lines using (7.4) and the definition (3.52).

To see that both coefficients vanish identically unless $d = \frac{|\mu|+|\nu|-|\lambda|}{n} \in \mathbb{Z}_{\geq 0}$, recall from the definition (7.39) and (3.52) that $\mathbf{Q}'_{\nu'}$ and $\mathbf{S}'_{\nu'}$ are polynomial in the a_i ’s and the z -dependence of the latter is given in (3.39). Thus, the overall power $\frac{|\mu|+|\nu|-|\lambda|}{n}$ of z in $\langle \lambda | \mathbf{Q}'_{\nu'} | \mu \rangle, \langle \lambda | \mathbf{S}'_{\nu'} | \mu \rangle$ must be a non-negative integer by definition. \blacksquare

Remark 7.4 *The notation chosen for the transition matrix is not coincidental. It is a generalisation of the modular \mathcal{S} -matrix of the Verlinde ring which is recovered when formally setting $t = 0$ in (7.25). In [40, Props 6.11 and 6.15, Def 6.13] it has been shown that the $\mathcal{S}(0)$ -matrix is the transition matrix from the Bethe vectors to the basis $\{|\lambda\rangle : \lambda \in \mathcal{A}_{k,n}^+\}$ and coincides with the famous Kac-Peterson formula. Thus, we can interpret (7.27) as ‘deformed Verlinde formula’.*

Formula (7.27) implies several obvious ‘symmetries’ of the matrix elements $\langle \lambda | \mathbf{Q}'_{\nu'} | \mu \rangle$ which we summarise in the following corollary.

Corollary 7.13 (symmetries) *Let $N_{\mu\nu}^\lambda(t) = \langle \lambda | \mathbf{Q}'_{\nu'} | \mu \rangle$ and set $z = 1$. Assume further that $\bar{t} = t$. Then we have the following identities:*

$$1. \ N_{\mu\nu}^\lambda(t) = N_{\nu\mu}^\lambda(t) \text{ and } \overline{N_{\mu\nu}^\lambda(t)} = N_{\mu\nu}^\lambda(t) = N_{\mu^* \nu^*}^{\lambda^*}(t)$$

2. Charge conjugation: $b_\lambda(t)N_{\mu\nu}^\lambda(t) = b_\nu(t)N_{\mu\lambda^*}^{\nu^*}(t)$
3. $N_{\mu n^k}^\lambda(t) = \delta_{\lambda\mu}$ and $N_{\lambda\mu}^{n^k}(t) = \delta_{\lambda\mu^*}/b_\lambda(t)$
4. Rotation invariance: $N_{\mu\nu}^\lambda(t) = N_{\text{rot}(\mu)\nu}^{\text{rot}(\lambda)}(t)$.

Proof. Note that invariance under complex conjugation, $\overline{N_{\mu\nu}^\lambda(t)} = N_{\mu\nu}^\lambda(t)$, follows simply from the definition (3.52) and observing that $\langle \lambda | a_i | \mu \rangle \in \mathbb{Z}[t]$ for all $i = 1, \dots, n$. The remaining identities in 1-2 are now trivial consequences of (7.21), $Q_\lambda = b_\lambda P_\lambda$ and $b_\lambda = b_{\lambda^*}$. For statement 3 simply use (7.19) and observe that $P_\emptyset = 1$, whence $N_{\mu n^k}^\lambda(t) = \sum_\sigma \mathcal{S}_{\mu\sigma}(t) \mathcal{S}_{\sigma\lambda}^{-1}(t) = \delta_{\mu\lambda}$. Now use statement 2 to obtain the second identity in 3. Finally, the last statement is a simply consequence of $P_{\text{rot}(\lambda)}(y; t) = e_k(y) P_{\text{rot}(\lambda)}(y; t)$, $b_\lambda = b_{\text{rot}(\lambda)}$ and that $e_k(\bar{y}) = e_k(y^{-1}) = e_k(y)^{-1}$. ■

We are now ready to prove the main result.

Proof. [Theorem 7.11] From the first and second property in the last Corollary it is now clear that the product (7.23) is commutative, $N_{\nu\mu}^\lambda(t) = \langle \lambda | \mathbf{Q}'_{\nu'} | \mu \rangle = \langle \lambda | \mathbf{Q}'_{\mu'} | \nu \rangle = N_{\mu\nu}^\lambda(t)$, and that $|n^k\rangle * |\lambda\rangle = |\lambda\rangle$. Associativity then easily follows from (3.49),

$$\begin{aligned} \lambda \otimes (\mu \otimes \nu) &= \sum_{\sigma \in \mathcal{A}_{k,n}^+} N_{\mu\nu}^\rho(t) \lambda \otimes \rho = \sum_{\rho, \sigma \in \mathcal{A}_{k,n}^+} N_{\lambda\rho}^\sigma(t) N_{\nu\mu}^\rho(t) \sigma \\ &= \sum_{\rho, \sigma \in \mathcal{A}_{k,n}^+} \langle \sigma | \mathbf{Q}'_{\nu'} \mathbf{Q}'_{\lambda'} | \mu \rangle \sigma = \sum_{\rho, \sigma \in \mathcal{A}_{k,n}^+} N_{\nu\rho}^\sigma(t) N_{\lambda\mu}^\rho(t) \sigma = (\lambda \otimes \mu) \otimes \nu, \end{aligned}$$

where for ease of notation we have denoted vectors simply by partitions. Thus, $(\mathfrak{F}_{n,k}, \otimes)$ is a unital, associative and commutative algebra. Since $\{\lambda : \lambda \in \mathcal{A}_{k,n}^+\}$ is a basis and the map $\lambda \mapsto \lambda^*$ simply amounts to a reordering of this basis, it is obvious that η is nondegenerate. One now easily checks with the help of (7.27) and $b_{\lambda^*}(t) = b_\lambda(t)$ that $\eta(\lambda, \mu) = \eta(\mu, \lambda)$ and

$$\eta(\lambda \otimes \mu, \nu) = N_{\lambda\mu}^{\nu^*}(t)/b_\nu(t) = N_{\mu\nu}^{\lambda^*}(t)/b_\lambda(t) = \eta(\lambda, \mu \otimes \nu)$$

according to the second property in Corollary 7.13. ■

Denote by $\mathbf{Q}_\lambda^{(k)}$ the restriction of \mathbf{Q}'_λ to the subspace $\mathcal{F}_k^{\otimes n}$ spanned by $\{|\lambda\rangle \mid \lambda \in \mathcal{A}_{k,n}^+\}$.

Corollary 7.14 (deformed fusion matrices) *Set $z = 1$ and consider the subalgebra $\subset \text{End } \mathfrak{F}_{n,k}$ generated by $\{\mathbf{Q}_{\lambda'}^{(k)} \mid \lambda \in \mathcal{A}_{k,n}^+\}$. Then the map $|\lambda\rangle \mapsto \mathbf{Q}_{\lambda'}^{(k)}$ is an algebra isomorphism. That is, we have for all $\mu, \nu \in \mathcal{A}_{k,n}^+$ the product expansion*

$$\mathbf{Q}_{\mu'}^{(k)} \mathbf{Q}_{\nu'}^{(k)} = \sum_{\lambda \in \mathcal{A}_{k,n}^+} N_{\mu\nu}^\lambda(t) \mathbf{Q}_{\lambda'}^{(k)}. \quad (7.30)$$

Proof. By definition of (7.23) and exploiting associativity we compute

$$\begin{aligned} \mathbf{Q}'_{\mu'} \mathbf{Q}'_{\nu'} | \rho \rangle &= |\mu\rangle \otimes (|\nu\rangle \otimes |\rho\rangle) = (|\mu\rangle \otimes |\nu\rangle) \otimes |\rho\rangle \\ &= \sum_{\lambda \in \mathcal{A}_{k,n}^+} N_{\mu\nu}^\lambda(t) |\lambda\rangle \otimes |\rho\rangle = \sum_{\lambda \in \mathcal{A}_{k,n}^+} N_{\mu\nu}^\lambda(t) \mathbf{Q}'_{\lambda'} | \rho \rangle \end{aligned}$$

for any $\rho \in \mathcal{A}_{k,n}^+$. Hence, the assertion follows. ■

Remark 7.5 The last presentation of the Frobenius algebra in $\text{End } \mathfrak{F}_{n,k}$ is particularly convenient to show that it specialises to the Verlinde ring when restricting the coefficients to $\mathbb{Z}[t]$. Namely, setting formally $t = 0$ in (3.41) by making the replacement $\hat{\mathcal{U}}_n^- \rightarrow \hat{\mathcal{U}}_n^- / \langle t \rangle$, one obtains a representation of the local affine plactic algebra [40, Def 5.4 and Prop 5.8] as mentioned earlier. Thus, the deformed fusion matrices specialise for $t = 0$ to

$$Q'_\lambda = S'_\lambda = \det(\mathbf{h}_{\lambda_i - i + j}) = \mathbf{s}_\lambda,$$

where $\mathbf{h}_r = \sum_{\lambda \vdash r} \mathbf{m}_\lambda$ is now the affine plactic homogeneous symmetric polynomial and \mathbf{s}_λ is the affine plactic Schur polynomial defined in [40, Def 5.15-6] and [41, Prop 4.1 and Prop 5.1]. The latter have been identified with the fusion matrices of the Verlinde ring [40, Theorem 6.18] and we can therefore conclude that the constant terms $N_{\mu\nu}^\lambda(0)$ and $K_{\nu,\lambda/d/\mu}(0)$ are the Wess-Zumino-Novikov-Witten fusion coefficients $\mathcal{N}_{\hat{\mu}\hat{\nu}}^{(k)\hat{\lambda}}$. The indices of the fusion coefficients are given via the following bijection $\lambda \mapsto \hat{\lambda} := \sum_{i=1}^n m_i(\lambda) \hat{\omega}_i$ between $\mathcal{A}_{k,n}^+$ and the set of integral dominant $\hat{\mathfrak{sl}}(n)$ weights at level k ,

$$P_{n,k}^+ = \left\{ \hat{\lambda} = \sum_{i=1}^n m_i \hat{\omega}_i \mid \sum_{i=1}^n m_i = k \right\}, \quad (7.31)$$

where the $\hat{\omega}_i$'s denote the affine fundamental weights.

The following proposition shows that our discussion of the Bethe ansatz in the previous section has the algebraic interpretation of computing its Peirce decomposition.

Proposition 7.15 (idempotents) The Bethe vectors (7.1) are the idempotents of the Frobenius algebra $(\mathfrak{F}_{k,n}, \otimes, \eta)$,

$$\mathbf{e}_\lambda \otimes \mathbf{e}_\mu = \delta_{\lambda\mu} \mathbf{e}_\lambda, \quad \mathbf{e}_\lambda := |\mathcal{S}_{0\lambda}|^2 \mathbf{b}_\lambda = \mathcal{S}_{0\lambda} \sum_{\mu \in \mathcal{A}_{k,n}^+} \mathcal{S}_{\lambda\mu}^{-1} |\mu\rangle. \quad (7.32)$$

Moreover, we have the following decomposition of the unit, $|n^k\rangle = \sum_{\lambda \in \mathcal{A}_{k,n}^+} \mathbf{e}_\lambda$. The dual Bethe vectors on the other hand obey

$$\Delta_{n,k} \mathbf{e}_\lambda^* = \mathbf{e}_\lambda^* \otimes \mathbf{e}_\lambda^*, \quad \mathbf{e}_\lambda^* := \mathbf{b}_\lambda^* / |\mathcal{S}_{0\lambda}|^2 = \sum_{\mu \in \mathcal{A}_{k,n}^+} \langle \mu | \frac{\mathcal{S}_{\mu\lambda}}{\mathcal{S}_{0\lambda}} \quad (7.33)$$

where $\Delta_{n,k} : \mathfrak{F}_{n,k} \rightarrow \mathfrak{F}_{n,k} \otimes \mathfrak{F}_{n,k}$ is the coproduct induced by η .

Proof. The first statement is a trivial consequence of (7.27),

$$\mathbf{e}_\lambda \otimes \mathbf{e}_\mu = \mathcal{S}_{0\lambda} \sum_{\nu \in \mathcal{A}_{k,n}^+} \mathcal{S}_{\lambda\nu}^{-1} Q'_{\nu'} \mathbf{e}_\mu = \mathcal{S}_{0\lambda} \sum_{\nu \in \mathcal{A}_{k,n}^+} \frac{\mathcal{S}_{\lambda\nu}^{-1} \mathcal{S}_{\nu\mu}}{\mathcal{S}_{0\mu}} \mathbf{e}_\mu = \delta_{\lambda\mu} \mathbf{e}_\lambda.$$

Thus, we have for any μ that $\mathbf{e}_\mu \otimes (\sum_\lambda \mathbf{e}_\lambda) = \mathbf{e}_\mu$. Since the $\{\mathbf{b}_\lambda\}$ and, hence, the $\{\mathbf{e}_\lambda\}$ are a basis it follows that for any $\mathbf{f} \in \mathfrak{F}_{n,k}$ we have $\mathbf{f} \otimes (\sum_\lambda \mathbf{e}_\lambda) = \mathbf{f}$. Setting $\mathbf{1} := |n^k\rangle$ and $\mathbf{1}' := \sum_\lambda \mathbf{e}_\lambda$ it follows that $\eta(\mathbf{1} - \mathbf{1}', \mathbf{f}) = \eta(\mathbf{1}, \mathbf{f} \otimes (\mathbf{1} - \mathbf{1}')) = 0$ for all $\mathbf{f} \in \mathfrak{F}_{n,k}$ and, therefore, $\mathbf{1} = \mathbf{1}'$ because η is non-degenerate.

To prove the second statement we explicitly compute the coproduct using the known facts about the structure of Frobenius algebras; see e.g. [39]. Let $\mathbf{m} : \mathfrak{F}_{n,k} \otimes \mathfrak{F}_{n,k} \rightarrow \mathfrak{F}_{n,k}$ be the regular representation or multiplication

map, $\mathfrak{m}(|\mu\rangle \otimes |\nu\rangle) = |\mu\rangle \otimes |\nu\rangle$ and $\mathfrak{m}^* : \mathfrak{F}_{n,k}^* \rightarrow \mathfrak{F}_{n,k}^* \otimes \mathfrak{F}_{n,k}^*$ its dual map. Then the co-product $\Delta_{n,k}$ is obtained via the following commutative diagram

$$\begin{array}{ccc} \mathfrak{F}_{n,k} & \xrightarrow{\Delta_{n,k}} & \mathfrak{F}_{n,k} \otimes \mathfrak{F}_{n,k} \\ \downarrow \Phi & & \downarrow \Phi \otimes \Phi \\ \mathfrak{F}_{n,k}^* & \xrightarrow{\mathfrak{m}^*} & \mathfrak{F}_{n,k}^* \otimes \mathfrak{F}_{n,k}^* \end{array} \quad , \quad (7.34)$$

where the Frobenius isomorphism $\Phi : \mathfrak{F}_{n,k} \rightarrow \mathfrak{F}_{n,k}^*$ is given by

$$\Phi : |\lambda\rangle \mapsto b_\lambda^{-1}(t) \langle \lambda^* | . \quad (7.35)$$

We claim that the coproduct in the basis $\{|\lambda\rangle\}$ is computed to

$$\Delta_{n,k} |\lambda\rangle = \sum_{\mu, \nu \in \mathcal{A}_{k,n}^+} \frac{b_\mu(t) b_\nu(t)}{b_\lambda(t)} N_{\mu\nu}^\lambda(t) |\nu\rangle \otimes |\mu\rangle . \quad (7.36)$$

Thus, we have in particular $(\Phi \otimes \Phi) \Delta_{n,k}(|\lambda\rangle)(|\nu\rangle \otimes |\mu\rangle) = b_\lambda^{-1} N_{\mu^* \nu^*}^\lambda$, where we have used once more that $b_{\mu^*} = b_\mu$. According to (7.34) this result has to match

$$\mathfrak{m}^* \circ \Phi(|\lambda\rangle)(|\nu\rangle \otimes |\mu\rangle) = b_\lambda^{-1} \langle \lambda^* | \mu \otimes \nu \rangle = b_\lambda^{-1} N_{\nu\mu}^{\lambda^*} .$$

That both results are indeed equal now follows from the properties in Corollary 7.13.

Identify the bra-vector $\langle \lambda |$ in $\mathfrak{F}_{n,k}^*$ with the ket-vector $b_\lambda |\lambda\rangle$ in $\mathfrak{F}_{n,k}$. Then a straightforward computation yields the last assertion,

$$\begin{aligned} \Delta_{n,k} \mathfrak{e}_\lambda^* &= \sum_{\mu, \nu, \rho} \frac{\mathcal{S}_{\mu\lambda}}{\mathcal{S}_{0\lambda}} N_{\nu\rho}^\mu b_\rho b_\nu |\rho\rangle \otimes |\nu\rangle \\ &= \sum_{\mu, \nu, \rho, \sigma} \frac{\mathcal{S}_{\mu\lambda}}{\mathcal{S}_{0\lambda}} \frac{\mathcal{S}_{\nu\sigma} \mathcal{S}_{\rho\sigma} \mathcal{S}_{\sigma\mu}^{-1}}{\mathcal{S}_{0\sigma}} b_\rho b_\nu |\rho\rangle \otimes |\nu\rangle \\ &= \sum_{\mu, \nu, \rho, \sigma} \frac{\mathcal{S}_{\nu\lambda} \mathcal{S}_{\rho\lambda}}{\mathcal{S}_{0\lambda}^2} b_\rho b_\nu |\rho\rangle \otimes |\nu\rangle = \mathfrak{e}_\lambda^* \otimes \mathfrak{e}_\lambda^* . \end{aligned}$$

Here we have used (7.27) in the second line. ■

Remark 7.6 *The explicit computation of the coproduct (7.36) ties the Frobenius algebra $\mathfrak{F}_{n,k}$ to our earlier discussion of cylindric skew Macdonald functions. Make the formal identification $|\lambda\rangle \mapsto Q'_{\lambda'}$ and $\langle \lambda | \mapsto P'_{\lambda'}$ but instead of taking the usual product (1.4) in the ring of symmetric functions define the fusion product $Q'_{\mu'} * Q'_{\nu'} := \sum_{\lambda \in \mathcal{A}_{k,n}^+} N_{\mu\nu}^\lambda(t) Q'_{\lambda'}$. Then the Frobenius coproduct yields*

$$\begin{aligned} \Delta_{n,k} P'_{\lambda'} &= \sum_{\mu, \nu \in \mathcal{A}_{k,n}^+} N_{\mu\nu}^\lambda(t) P'_{\nu'} \otimes P'_{\mu'} \\ &= \sum_{\mu \in \mathcal{A}_{k,n}^+, d \geq 0} P'_{\lambda'/d/\mu'} \otimes P'_{\mu'} . \end{aligned}$$

This links the Frobenius algebra $\mathfrak{F}_{n,k}$ to the partition function of the statistical mechanics model with transfer matrix (3.46).

In principle, we can compute the structure constants $N_{\mu\nu}^\lambda(t)$ of the Frobenius algebra $\mathfrak{F}_{n,k}$ from the representation (3.39) employing (7.23) and the definition (3.52). An alternative is to use our description of the coordinate ring $\mathbb{k}[\mathbf{V}_{k,n}]$ in the previous section.

Theorem 7.16 (restricted Hall algebra) *The map $|\lambda\rangle \mapsto [P_\lambda]$ defines for $z = 1$ an algebra isomorphism $\mathfrak{F}_{n,k} \cong \mathbb{k}[\mathbf{V}_{k,n}]$. That is, the coefficients in the product expansions*

$$[P_{\tilde{\mu}}][P_{\tilde{\nu}}] := [P_{\tilde{\mu}}P_{\tilde{\nu}}] = \sum_{\lambda \in \mathcal{A}_{k,n}^+} \tilde{N}_{\mu\nu}^\lambda(t) [P_\lambda], \quad (7.37)$$

and

$$[P_{\tilde{\mu}}][s_{\tilde{\nu}}] := [P_{\tilde{\mu}}s_{\tilde{\nu}}] = \sum_{\lambda \in \mathcal{A}_{k,n}^+} K_{\nu,\lambda/d/\mu}(t) [P_\lambda] \quad (7.38)$$

coincide with the expansion coefficients of the cylindric skew Macdonald functions (6.8),

$$\tilde{N}_{\mu\nu}^\lambda(t) = N_{\mu\nu}^\lambda(t) = \langle \lambda | \mathbf{Q}'_{\tilde{\nu}} | \mu \rangle \quad \text{and} \quad K_{\nu,\lambda/d/\mu}(t) = \langle \lambda | \mathbf{S}'_{\tilde{\nu}} | \mu \rangle. \quad (7.39)$$

Proof. The proof rests once more on the existence of an eigenbasis, Theorem 7.8, and the expression (7.27). Namely, using (7.19) we have

$$\begin{aligned} N_{\mu\nu}^\lambda(t) &= \sum_{\sigma \in \mathcal{A}_{k,n}^+} \frac{P_{\tilde{\mu}}(y_\sigma; t) P_{\tilde{\nu}}(y_\sigma; t) \mathcal{S}_{\sigma\lambda}^{-1}(t)}{\|\mathbf{b}_\sigma\|} \\ &= \sum_{\rho, \sigma \in \mathcal{A}_{k,n}^+} \frac{\tilde{N}_{\mu\nu}^\rho(t) P_{\tilde{\rho}}(y_\sigma; t) \mathcal{S}_{\sigma\lambda}^{-1}(t)}{\|\mathbf{b}_\sigma\|} \\ &= \sum_{\rho \in \mathcal{A}_{k,n}^+} \tilde{N}_{\mu\nu}^\rho(t) \sum_{\sigma \in \mathcal{A}_{k,n}^+} \mathcal{S}_{\rho\sigma}(t) \mathcal{S}_{\sigma\lambda}^{-1}(t) = \tilde{N}_{\mu\nu}^\lambda(t). \end{aligned} \quad (7.40)$$

The second line employs Lemma 7.4 which ensures that the expansion of the product $P_{\tilde{\mu}}(y_\sigma; t) P_{\tilde{\nu}}(y_\sigma; t)$ equals the expansion of $[P_{\tilde{\mu}}P_{\tilde{\nu}}]$ in $\mathbb{k}[\mathbf{V}_{k,n}]$. The second assertion follows from an analogous computation. ■

Recall that $Q_{(r)} = g_r$ and $P_{(1^r)} = e_r$, then a direct consequence of our earlier computations and the last theorem is the following obvious corollary which links the transfer matrices (3.42) and (3.46) to the coordinate ring.

Corollary 7.17 (Pieri rules in the quotient) *Let $\mu \in \mathcal{A}_{k,n}^+$ and $0 \leq r < n$, $0 \leq r' \leq k$. Then we have the following modified Pieri rules in the coordinate ring $\mathfrak{R}_{n,k}[z] := \mathbb{k}[\mathbf{V}_{k,n}] \otimes_{\mathbb{k}} \mathbb{k}[z]$,*

$$[g_r P_{\tilde{\mu}}] = \sum_{\lambda \in \mathcal{A}_{k,n}^+} \langle \lambda | e_r | \mu \rangle [P_\lambda] = \sum_{\substack{\lambda/d/\mu=(r), \\ \lambda \in \mathcal{A}_{k,n}^+}} z^d \Phi_{\lambda/d/\mu}(t) [P_\lambda], \quad (7.41)$$

$$[e_{r'} P_{\tilde{\mu}}] = \sum_{\lambda \in \mathcal{A}_{k,n}^+} \langle \lambda | g_{r'} | \mu \rangle [P_\lambda] = \sum_{\substack{\lambda/d/\mu=(1^{r'}), \\ \lambda \in \mathcal{A}_{k,n}^+}} z^d \Psi'_{\lambda'/d/\mu'}(t) [P_\lambda]. \quad (7.42)$$

Remark 7.7 (Open boundary conditions) *Note that either of the Pieri rules (7.41) and (7.42) fixes the product in $\mathfrak{R}_{n,k}[z]$. Setting $z = 0$ and $t = -1$ the quotient $\mathfrak{R}_{n,n}[z]/\langle z, t + 1 \rangle$ is isomorphic to the cohomology ring of the orthogonal Grassmannian $OG(n, 2n)$ in the basis of P -polynomials. The Pieri rule for Q -polynomials coincides with the cohomology ring of the Lagrangian Grassmannian $LG(n - 1, 2n - 2)$. For general z we obtain a deformation of these cohomology rings which is different from the usual quantum cohomology [10].*

7.3.1 Algorithm to compute deformed fusion coefficients

We demonstrate on an explicit example how the expansion coefficients (6.16) and (6.17) can be computed in the coordinate ring $\mathbb{K}[\mathbf{V}_{k,n}]$. The general procedure can be described as follows: first compute the normal product expansion $P_{\tilde{\mu}}P_{\tilde{\nu}} = \sum_{\lambda} f_{\mu\nu}^{\lambda}(t)P_{\lambda}$ in $\mathbb{Z}[t][e_1, \dots, e_k]$, which is possible since explicit formulae for computing the coefficients $f_{\mu\nu}^{\lambda}(t)$ are known; see [53]. In the second step rewrite those P_{λ} with λ outside the fundamental region $\tilde{\mathcal{A}}_{k,n}^{+}$ in terms of $P_{\tilde{\lambda}}$'s with $\tilde{\lambda} \in \tilde{\mathcal{A}}_{k,n}^{+}$ using first the affine (7.18) and then the non-affine straightening rules (7.15), (7.16). Collecting coefficients of the individual terms one obtains the expansion in $\mathbb{K}[\mathbf{V}_{k,n}]$ and, hence, $N_{\mu\nu}^{\lambda}(t)$.

Example 7.2 *Set $n = 4$, $k = 3$ and $z = 1$. Consider the partitions $\lambda = (3, 2, 1)$ and $\mu = (4, 3, 1)$. We exploit the fact that $f_{\lambda\mu}^{\nu}(t) \equiv 0$ unless $f_{\lambda\mu}^{\nu}(0) = c_{\lambda\mu}^{\nu} \neq 0$ where $c_{\lambda\mu}^{\nu}$ is the Littlewood-Richardson coefficient. Perform the Littlewood-Richardson algorithm, one finds that for all partitions ν of length $\leq k$ the nonzero coefficients are*

$$c_{\lambda\mu}^{(7,5,2)} = c_{\lambda\mu}^{(7,4,3)} = c_{\lambda\mu}^{(6,6,2)} = c_{\lambda\mu}^{(6,4,4)} = c_{\lambda\mu}^{(5,5,4)} = 1, \quad c_{\lambda\mu}^{(6,5,3)} = 2.$$

With the help of a computer one then calculates the following expansion coefficients in the corresponding product of Hall-Littlewood polynomials,

$$f_{\lambda\mu}^{(7,5,2)} = f_{\lambda\mu}^{(7,4,3)} = 1, \quad f_{\lambda\mu}^{(6,6,2)} = f_{\lambda\mu}^{(6,4,4)} = f_{\lambda\mu}^{(5,5,4)} = 1 + t, \quad f_{\lambda\mu}^{(6,5,3)} = 2 - t^2.$$

Applying the straightening rules (7.17) and (7.15) one finds

$$\begin{aligned} P_{(7,5,2)} &= tP_{(3,2,1)}, & P_{(7,4,3)} &= [2]P_{(4,3,3)}, & P_{(6,6,2)} &= [3]P_{(2,2,2)}, \\ P_{(6,5,3)} &= P_{(3,2,1)}, & P_{(6,4,4)} &= P_{(4,4,2)}, & P_{(5,5,4)} &= P_{(4,1,1)}. \end{aligned}$$

Thus, after removing all n -rows we arrive at the expansion

$$\begin{aligned} P_{(3,2,1)}P_{(4,3,1)} &= (2 + t - t^2)P_{(3,2,1)} + (1 + t)(1 + t + t^2)P_{(2,2,2)} \\ &\quad + (1 + t)(P_{(3,3,0)} + P_{(1,1,0)} + P_{(2,0,0)}). \end{aligned}$$

From this computation we thus obtain the following t -deformed fusion coefficients

$$\begin{aligned} N_{\lambda\mu}^{(3,2,1)}(t) &= 2 + t - t^2, & N_{\lambda\mu}^{(2,2,2)}(t) &= (1 + t)(1 + t + t^2), \\ N_{\lambda\mu}^{(4,3,3)}(t) &= N_{\lambda\mu}^{(4,1,1)}(t) = N_{\lambda\mu}^{(4,4,2)}(t) &= 1 + t. \end{aligned}$$

Setting $t = 0$ one can verify, using the Verlinde formula or the Kac-Walton algorithm, that one obtains the correct fusion coefficients of the $\widehat{\mathfrak{sl}}(n)_k$ -Verlinde algebra.

So far we have focussed on the matrix elements $N_{\mu\nu}^\lambda(t) = \langle \lambda | \mathbf{Q}'_{\nu'} | \mu \rangle$. However, the definition (3.52) expresses the latter in terms of the matrix elements $K_{\nu, \lambda/d/\mu}(t) = \langle \lambda | \mathbf{S}'_{\nu'} | \mu \rangle$. Sample computations of the latter lead to the following observation.

Conjecture 7.18 *Let $\lambda, \mu \in \mathcal{A}_{k,n}^+$ and $\nu \in \tilde{\mathcal{A}}_{k,n}^+$. The expansion coefficients (matrix elements) $\langle \lambda | \mathbf{S}'_{\nu'} | \mu \rangle$ are always polynomials with non-negative coefficients.*

This conjecture has been numerically verified for several examples.

Example 7.3 *Choose $n = k = 5$ and set $\mu = (3, 2, 2, 1, 1)$, $\nu = (4, 3, 1)$. Then the table below lists all $\lambda \in \mathcal{A}_{k,n}^+$ for which the matrix element $\langle \lambda | \mathbf{S}'_{\nu'} | \mu \rangle$ is nonvanishing.*

λ	$\langle \lambda \mathbf{S}'_{\nu'} \mu \rangle$
4, 2, 2, 2, 2	$1 + 3t + 6t^2 + 8t^3 + 8t^4 + 6t^5 + 3t^6 + t^7$
4, 3, 2, 2, 1	$2 + 9t + 16t^2 + 14t^3 + 6t^4 + t^5$
4, 3, 3, 1, 1	$1 + 6t + 13t^2 + 14t^3 + 8t^4 + 2t^5$
4, 4, 2, 1, 1	$1 + 6t + 11t^2 + 10t^3 + 3t^4$
2, 2, 1, 1, 1	$1 + 3t + 4t^2 + 3t^3 + t^4$
3, 1, 1, 1, 1	$1 + 2t + 3t^2 + 3t^3 + 2t^4 + t^5$
5, 4, 3, 3, 2	$1 + 7t + 20t^2 + 31t^3 + 29t^4 + 17t^5 + 6t^6 + t^7$
5, 4, 4, 2, 2	$1 + 6t + 17t^2 + 24t^3 + 20t^4 + 9t^5 + 2t^6$
5, 2, 2, 2, 1	$2 + 6t + 9t^2 + 7t^3 + 3t^4$
5, 3, 2, 1, 1	$3 + 8t + 9t^2 + 3t^3$
5, 4, 1, 1, 1	$1 + 3t + 4t^2 + 3t^3 + t^4$
5, 5, 3, 2, 2	$2 + 8t + 16t^2 + 17t^3 + 10t^4 + 3t^5$
5, 5, 3, 3, 1	$1 + 5t + 10t^2 + 10t^3 + 5t^4 + t^5$
5, 5, 4, 2, 1	$1 + 5t + 8t^2 + 6t^3 + 2t^4$

Note that the constant term for each listed polynomial coincides with the fusion coefficient.

Remark 7.8 *Let $\lambda, \mu \in \mathcal{A}_{k,n}^+$ and $\nu \in \tilde{\mathcal{A}}_{k,n}^+$. Then for $dn = |\mu| + |\nu| - |\lambda| = 0$ the matrix elements $N_{\mu\nu}^\lambda(t) = \langle \lambda | \mathbf{Q}'_{\nu'} | \mu \rangle$ and $K_{\nu, \lambda/d/\mu}(t) = \langle \lambda | \mathbf{S}'_{\nu'} | \mu \rangle$ specialise to the known polynomials*

$$f_{\mu\nu}^\lambda(t) = \langle Q_{\lambda/\mu}, P_\nu \rangle_t = \langle Q_\lambda, P_\mu P_\nu \rangle_t$$

and

$$K_{\nu, \lambda/\mu}(t) = \sum_{\rho \in \tilde{\mathcal{A}}_{k,n}^+} K_{\nu\rho}(t) f_{\rho\mu}^\lambda(t) = \langle S_\nu, Q_{\lambda/\mu} \rangle_t = \langle Q_\lambda, P_\mu s_\nu \rangle_t,$$

respectively. Here S_λ is the dual Schur function with respect to the Hall inner product; see (2.30).

Example 7.4 *Let $\nu = (4, 3, 1, 0, 0)$, $\mu = (3, 2, 2, 1, 1)$, $\lambda = (5, 5, 3, 2, 2)$ then*

$$f_{\mu\nu}^\lambda(t) = 2 + 3t + t^2 - t^3 - t^4$$

and

$$K_{\nu, \lambda/\mu}(t) = 2 + 8t + 16t^2 + 17t^3 + 10t^4 + 3t^5$$

With the help of a computer one finds the following values for the Hall and Kostka-Foulkes polynomials from the known formulae in [53, Chapter III.6],

ρ	$f_{\rho\mu}^\lambda(t)$	$K_{\nu\rho}(t)$
4, 3, 1	$2 + 3t + t^2 - t^3 - t^4$	1
4, 2, 2	$1 + t$	t
4, 2, 1, 1	$1 + 2t + t^2$	$t + t^2$
4, 1, 1, 1, 1	0	$t^3 + t^4 + t^5$
3, 3, 2	$1 + t - t^2 - t^3$	$t + t^2$
3, 3, 1, 1	$2 + 2t - 2t^3 - t^4$	$t + 2t^2 + t^3$
3, 2, 2, 1	$1 + 2t + t^2$	$2t^2 + 2t^3 + t^4$
3, 2, 1, 1, 1	$1 + t$	$t^2 + 2t^3 + 3t^4 + 2t^5 + t^6$
2, 2, 2, 2	0	$t^3 + t^4 + 2t^5 + 2t^6 + t^7$
2, 2, 2, 1, 1	0	$t^3 + 3t^4 + 3t^5 + 3t^6 + 2t^7 + t^8$

8 Conclusions

There has been recently a lot of attention on integrable quantum many-body systems, such as the nonlinear quantum Schrödinger model, in connection with the infrared limit of four-dimensional supersymmetric $N = 2$ gauge theories [57]. It has been observed that the quantum Hamiltonians can be identified with operators in the chiral ring, that is operators that are annihilated by one chiral half of the supercharges. Moreover, it has been asserted that the Bethe ansatz equations describe the vacua of the gauge theory and that their solutions, the Bethe roots, have the interpretation of coordinates on the moduli space of these vacua. It has been further argued that the vacua should correspond to the states of a 2D topological quantum field theory. The latter are known to be in correspondence with commutative Frobenius algebras [39].

In light that the quantum integrable model investigated in this article is a discrete version of the quantum nonlinear Schrödinger model [73] our findings summarised in the following table confirm the connection between quantum integrable systems and two-dimensional topological quantum field theories.

quantum integrable model	quantum Hamiltonians	Bethe vectors	Bethe ansatz equations
Frobenius algebra	generators	idempotents	coordinate ring presentation

Table 8.1. Dictionary between commutative Frobenius algebras and quantum integrable models.

Moreover, our discussion highlights the central role of the eigenbasis of the quantum Hamiltonians, the so-called Bethe vectors: they provide the algebra isomorphism between the subalgebra in $\text{End } S^k(V)$ generated by the deformed fusion matrices Q'_ν 's and the quotient of the spherical Hall algebra. The latter is the coordinate ring we discussed in Section 7 and according to the above correspondence it should have the interpretation of the moduli space of vacua of a quantum field theory.

In this context we note that the Verlinde algebra can be understood in terms of a purely topological construction using so-called equivariant K -theory [21]. Frobenius algebra deformations of the Verlinde algebra have been suggested in [71] and it would be interesting to see if these constructions are related.

The discussion here is very much motivated along similar lines as investigations of the so-called Bethe algebra for the Gaudin and related models. There it has been shown that the Bethe algebra related to Yangians describes the equivariant cohomology of flag manifolds [61]. In contrast the discussion in [40–42] shows that the representations of the Bethe algebra related to the quantum group $U_q\widehat{\mathfrak{gl}}(n)$ in the crystal limit, $q = 0$, are identical to the WZNW fusion ring and the small *quantum* cohomology ring of the Grassmannian. The present work extends this discussion to the case when $q \neq 0$.

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